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INFORMATION THEORY

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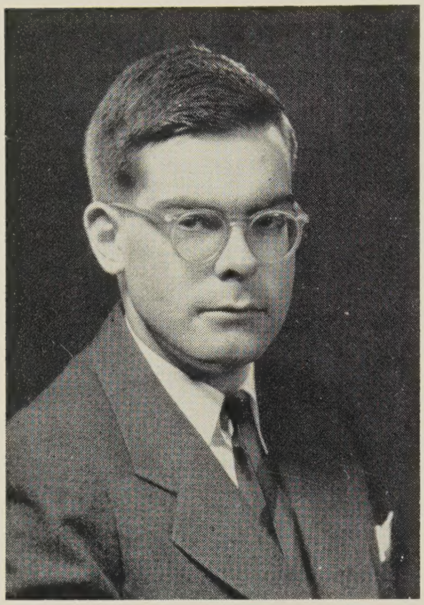

Robert Price

Robert Price (S'48–A'54) was born on July 7, 1929, in West Chester, Pennsylvania. He received the A.B. degree in physics from Princeton University, Princeton, N. J., in 1950, and the Sc.D. degree in electrical engineering from the Massachusetts Institute of Technology, Cambridge, Mass., in 1953.

His doctoral study was started at the Research Laboratory of Electronics of M.I.T. and continued later at the Lincoln Laboratory, Lexington, Mass. While at the Research Laboratory of Electronics, he held an Industrial Fellowship. His dissertation was concerned with the application of modern statistical methods to the problems of communicating over a channel characterized by time-varying multipath and noise. Upon graduation from M.I.T., he spent a year as a Fulbright Fellow, engaged in radio-astronomy at the Radiophysics Division of the Commonwealth Scientific and Industrial Research Organization in Sydney, Australia. Returning to the M.I.T. Lincoln Laboratory in 1954, he again became interested in multipath channels, and is a co-originator of the Rake antimultipath communication system. Recently he has been concerned with the Venus radar experiment.

His interests lie in both the theoretical and experimental aspects of statistical communication theory, and he has written on detection theory, signal statistics, and ionospheric measurement techniques, in addition to papers dealing with operational systems.

Dr. Price is a member of Phi Beta Kappa, Sigma Xi, and the Franklin Institute. He also recently has been elected a member of the U.S.A. Subcommission 6.1 of URSI.

The Search for Truth

ROBERT PRICE*

THERE is a tendency in our field to be apologetic that modern statistical methods have not yet made the bold impact on communication technology that as predicted with enthusiasm a few years ago. If we are ndid, we must admit that today it is a rare communi- ation system, either in the field or presently in fabrication, at exhibits a "new look" directly attributable to our fforts. One need only scan the TRANSACTIONS of our ster Professional Group on COMMUNICATIONS SYSTEMS to e that contemporary system design is drawn in the main om the pre-Wiener era.

To be sure, there are a number of techniques in current se that, while dating back some years, provide good pproximations to statistical optimality. The improve- ent in noisy signals achieved through "common sense" esign of filters in the frequency domain, the use of pulse ode modulation to obtain reliable digital communication, nd the reduction of digital system error rates realized hrough the use of matched filters, are examples of current ractice compatible with the precepts of our theories. It onfidently can be claimed further that modern statistical nalysis has been able, in many cases, to show the upper mits of attainment, and thereby has forestalled untold usings and profitless experiment. In this respect the ecognition of the governing roles of irreducible error, hannel capacity, and energy-to-noise-density ratio has een of inestimable value.

These accomplishments are of an essentially negative haracter, however, and as such we cannot take undue ride in them. What we really should look forward to, or o we are told, are the coming positive applications of tatistical theory. When certain difficult problems are nally resolved, there will appear systems, or at least ajor parts of systems, whose performance will be radi- ally improved over that we know today. Certainly there ill be at least one advance of this magnitude when resent studies on practical coding and decoding schemes ave progressed to the point where the Fundamental heorem of Information Theory can be realized in an perational system.

While we are waiting to revolutionize the communi- ation field, however, I would like to argue that the nsights and glimpses of fundamental truths already ained are of great intrinsic merit. A philosopher might ontend that these truths are our only real achievements, and that what improvement we may introduce in this or that communication system is of relatively lesser significance in the eternal scheme of things.

In illustration of such fundamental truths, I would first select the truly remarkable Fundamental Theorem. Shannon's discovery that errors can be made negligible while preserving a positive information rate over a noisy channel is certainly a major triumph. A second basic concept, which underlies the Fundamental Theorem but appears to be of significance in itself, is that of informational entropy. Although informational entropy does not strictly obey the conservational and invariance laws usually expected of a basic entity, no rival measure has come forth to dispute its preeminence. Both here and in Russia, the entropy measure has been found closely linked with filtering theory, and entropy could conceivably play an important role in nonlinear filtering studies. It would seem, in any case, that the last word has not yet been said about the properties of informational entropy. A third deep insight resides in Woodward's observation that all the information in an effect about its possible causes is contained in the set of *a posteriori* probabilities relating the effect back to the causes. This simple notion is a powerful one, for it gives understanding of why the likelihood ratio is at the root of detection and decision theory.

We may hope that yet further prime truths await discovery, and that the future will see better unity between our sub-disciplines of filtering theory, information theory, detection theory, and the analysis of signal statistics. It seems to me that there is just as much need for fresh ideas in our field as there is for the successful practical application of our present store of knowledge.

What I am pleading here is nothing more than the importance of basic research, and the dignity of knowledge for its own sake. Although statistical communication theory was largely inspired in its origins by the hope of improving communication and related systems, there is no reason why it must be tied entirely to its early purpose. Needless to say, I am by no means advocating the abandonment of the communication industry; on the contrary, cross-fertilization between pure and applied research in our field is bound to reap dividends for both.

Lest someone expect a sequel to appear under the title of "Information Theory, Photosynthesis, and Religion," I want to emphasize that truly basic studies are thorough and well-grounded. Their rigor is not sapped by casual overextension into a conglomeration of neighboring fields.

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At the other extreme, basic research is not served by tedious examination of one very restricted special case after another, unless it be with the purpose of seeking a common unity.

In conclusion, then, past accomplishments can be regarded with satisfaction but not with complacence. Ther will always be the continuing obligation to improv practical systems. But let us never fail to keep delvin for those deeper, unifying secrets that surely yield th ultimate rewards.

Experimental Results in Sequential Detection*

H. BLASBALG†

Summary—The main body of this paper reports on experi-ental results in sequential detection. In particular, it is shown at the Wald Theory of Sequential Analysis agrees well with periment for the important case of Bernoulli Detection even when e excess over the boundaries at the termination of an experiment neglected. The design of the experiments, as well as the experi-ental apparatus, are also discussed. Experimental curves of the perating Characteristic (OC) and Average Sample Number SN) Functions for several sets of parameters are given.

A publication relative to the main body of this paper[1] is sum-arized. The results of this publication are used in the Addendum, study the resonant properties of the exponential class of se-ential detectors. The practical use of these detectors for param-er estimation is discussed.

I. Introduction

THE experimental results described here serve primarily two purposes. First, it is well known that for the discrete independent case the charac-ristics which describe the performance of a sequential etector are only approximate. The errors are introduced ecause the excess over the decision boundaries at the rmination of an experiment is neglected. This introduces rors in the OC Function as well as in the ASN Function. 'ald[2] has obtained upper and lower bounds on these nctions which are expected to be quite satisfactory for reshold signals, for example, when a large sample umber is required, on the average, for the termination of n experiment. It is our aim to study these characteristics xperimentally.

The conventional way of constructing a sequential xperiment is to measure the ASN and OC functions for arious sets of threshold parameters $(\theta_0, \theta_1, \alpha, \beta)$ as a nction of the parameter θ corresponding to a one arameter family of distributions. The four-dimensional arameter point $(\theta_0, \theta_1, \alpha, \beta)$ defines a sequential test[2] f strength (α, β). The one parameter family of random ariables is then generated by varying the parameter θ. ach random variable in the family generates one point n the OC Function, $L(\theta)$, and on the corresponding SN Function. However, it has been shown by the uthor[1] that for a very general class of one parameter milies of distributions of practical and physical interest, sequential test is defined by a set of three transformations $a'(\theta_0, \theta_1, \alpha, \beta\}$, $b'(\theta_0, \theta_1, \alpha, \beta)$, $c'(\theta_0, \theta_1)\}$. Thus, there is a degeneracy in sequential analysis with respect to this class of one parameter families of distributions. Hence, there exists an infinity of parameter points $(\theta_0, \theta_1, \alpha, \beta)$ all of which have the same sequential detector defined by a three-dimensional parameter point (a', b', c'). It is therefore appropriate to measure the ASN and OC Functions as a function of θ for a given set (a', b', c'). Since the latter point is of general interest and of particular importance in this paper, we will summarize very briefly the results found earlier.[1]

For the class of distribution functions given by

$$dP(x, \theta) = \exp\,[r(\theta)A(x) + s(\theta)B(x)]\,dw(x), \tag{1}$$

it is shown that a set of three transformations

$$a' = \frac{\log\dfrac{\xi}{1-\xi} + \log\dfrac{1-\beta}{\alpha}}{r(\theta_1) - r(\theta_0)}, \tag{2}$$

$$b' = \frac{\log\dfrac{\xi}{1-\xi} + \log\dfrac{\beta}{1-\alpha}}{r(\theta_1) - r(\theta_0)}, \tag{3}$$

$$c' = \frac{s(\theta_1) - s(\theta_0)}{r(\theta_1) - r(\theta_0)}, \tag{4}$$

completely define the sequential probability ratio test for testing the hypothesis H_0 that $\theta \leq \theta_0$ against H_1 that $\theta \geq \theta_1 (\theta_1 > \theta_0)$. When the observer specifies the threshold parameters θ_0 and θ_1, corresponding to the hypothesis H_0 and H_1 and the strength (α, β) of the test, he specifies the three transformations and hence the sequential test. However, there is an infinity of parameter points $(\theta_0, \theta_1, \alpha, \beta)$ which satisfy the same transformations and hence the same sequential test.

For a given *a priori* probability ξ, it is known that the sequential probability ratio test is optimum[3] in the sense that the ASN Function at the parameter point $(\theta_0, \theta_1, \alpha, \beta)$ is less or equal to the ASN Function for any other sequential test. Since these parameter points specify the (a', b', c') transformations of the sequential probability ratio test, this test is also optimum for a given set (a', b', c'). Since for a given *a priori* probability ξ and a given set (a', b', c') there exists an infinity of parameter points $(\theta_0, \theta_1, \alpha, \beta)$ which satisfy the transformations,[1,4] we conclude that a given sequential test is optimum at an infinity of parameter points $(\theta_0, \theta_1, \alpha, \beta)$ which satisfy the given transformations.

* Manuscript received by the PGIT, June 3, 1958. This re-earch was supported by the United States Air Force through the ffice of Scientific Research of the Air Research and Development ommand.

† Electronic Communications, Inc., Timonium, Md.

[1] H. Blasbalg, "Transformation of the fundamental relation-ips in sequential analysis," *Annals of Math Statistics*, vol. 28, p. 1024–1028; December, 1957.

[2] A. Wald, "Sequential Analysis," John Wiley and Sons, Inc., ew York, N. Y.; 1947.

[3] D. Blackwell, and M. A. Girshick, "Theory of Games and Statistical Decisions," John Wiley and Sons, Inc., New York, N. Y.; 1954.

[4] M. A. Girshick, "An extension of the optimum property of the sequential probability ratio test," *Annals of Math Statistics*, vol. 29, pp. 288–290; March, 1958.

In a coordinated paper,[5] it is shown that the one parameter family of distributions in (1) is, from a practical point of view, the only family which exhibits this degeneracy with respect to sequential analysis. This family of distributions includes the very important and well-known exponential class of distributions which contains the Gaussian, Rayleigh, Bernoulli, Poisson, etc.

The optimum decision regions for this general class in terms of the (a', b', c') transformations are given by the simple relationships,

$$\sum_{i=1}^{n} A(x_i) + c' \sum_{i=1}^{n} B(x_i) \geq a' \rightarrow H_1 \tag{5}$$

and,

$$\sum_{i=1}^{n} A(x_i) + c' \sum_{i=1}^{n} B(x_i) \leq b' \rightarrow H_0. \tag{6}$$

The arrows indicate the hypotheses to which the decision regions belong.

The OC Function is given by the parametric equations

$$L(u) = \frac{u^{a'} - 1}{u^{a'} - u^{b'}} \tag{7}$$

$$E_\theta[u^{A(x)+c'B(x)}] = 1, \tag{8}$$

and the ASN Function is given by

$$E_\theta(n) = \frac{b'L(u) + a'[1 - L(u)]}{E_\theta\{A(x)\} + c'E_\theta\{B(x)\}}. \tag{9}$$

At the value $\theta = \theta'$ for which,

$$E_\theta\{A(x)\} + c'E_\theta\{B(x)\} = 0, \tag{10}$$

which corresponds to $u = 1$, we have,

$$L(1) = \frac{a'}{a' + |b'|} \tag{11}$$

and,

$$E_{\theta'}(n) = \frac{a'|b'|}{E_{\theta'}\{[A(x) + c'B(x)]^2\}}. \tag{12}$$

The parameter u ranges over the positive half of the real line $(0 \leq u \leq \infty)$, and E_θ is the conditional expected value operator of the random variable x.

Since the one parameter family of Bernoulli Distributions belongs to the general class for which these results apply and since this class is of practical importance in detection problems,[6] we will measure the OC and ASN Functions for Bernoulli Detection. In fact it was shown by the author[6] that the Bernoulli case exhibits this degeneracy. The general expressions for the bounds on these functions obtained by Wald[7] can be easily specialized for the Bernoulli case. In order to obtain the characteristics, we are forced into the practical design of a Bernoulli Sequential Detector for which the theory is developed.[6] For the sake of simplicity, we will investigate the OC and ASN Functions for the case $a' = -b'$. This corresponds to the case where the type I and type II errors α and β are equal.

[5] L. J. Savage, "When different pairs of hypotheses have the same family of likelihood-ratio test regions," *Annals of Math Statistics*, vol. 28, pp. 1028–1032; December, 1957.

[6] H. Blasbalg, "The relationship of sequential filter theory to information theory and its application to the detection of signals in noise by Bernoulli trials," IRE Trans. on Information Theory, vol. IT-3, pp. 122–131; June, 1957.

[7] A. Wald, *op. cit.*, pp. 161–179.

II. Design of an Experimental Bernoulli Sequential Detector

For the Bernoulli case we have slightly modified the choice of the transformations in order to simplify the desired form for the decision regions. These transformations are related in a very simple way to the general transformations given in (2), (3), and (4). It is easy to show that the relationship between (a', b', c') and the (a, b, c) transformations used here is

$$a' = \frac{a}{c+1},$$

$$b' = \frac{b}{c+1},$$

$$c' = \frac{1}{c+1}.$$

In an earlier paper[6] it was shown that the decision regions of the Bernoulli Sequential Detector are given by

$$k_n \leq \frac{b}{c+1} + \frac{n}{c+1} \rightarrow H_0 \tag{13}$$

$$k_n \geq \frac{a}{c+1} + \frac{n}{c+1} \rightarrow H_1 \tag{14}$$

where

k_n = number of ones measured in n Bernoulli trials

n = total number of ones and zeros measured.

The constants (a, b, c) are given as,

$$a = \frac{\log \dfrac{1-\beta}{\alpha}}{\log \dfrac{1-p_0}{1-p_1}}, \tag{15}$$

$$b = \frac{\log \dfrac{\beta}{1-\alpha}}{\log \dfrac{1-p_0}{1-p_1}}, \tag{16}$$

$$c = \frac{\log \dfrac{p_1}{p_0}}{\log \dfrac{1-p_0}{1-p_1}}, \tag{17}$$

where,

p_1 = probability of measuring a "one" when the hypothesis H_1 is true.

p_0 = probability of measuring a "one" when the hypothesis H_0 is true.

α = probability of accepting H_1 when H_0 is true.

β = probability of accepting H_0 when H_1 is true.

In (13) and (14), if we let $n = k_n + l_n$ where l_n is the umber of zeros in n trials, then the decision regions take he form,

$$\left.\begin{aligned} k_n c - l_n + \mid b \mid \leq 0 \rightarrow H_0 \\ k_n c - l_n - a \geq 0 \rightarrow H_1 \end{aligned}\right\}. \tag{18}$$

n our case, we will make (a, b, c) integers. Since k_n and are also integers, the physical realization of the Bernoulli Detector is a preset reversible binary counter. The eversible binary counter is preset to the number $|b|$. Each time a one is measured c ones are read into the ounter in the forward direction. Each time a zero is leasured a single count is fed into the binary counter in he reverse direction. If the counter returns to zero efore reaching the threshold $a + |b|$, the hypothesis H_0 is accepted that the parameter of the Bernoulli Distri-ution is $p \leq p_0$. On the other hand, if the counter reaches he number $a + |b|$ prior to reaching zero, the hypothesis H_1 is accepted that the Bernoulli parameter is $p \geq p_1$. rom (18) it is seen that whenever H_0 is accepted there is o excess over the boundaries at the termination of an xperiment with the acceptance of H_0. Thus for the range f parameters $p \leq p_0$ which almost always leads to the cceptance of H_0, we expect very close agreement between heory and experiment.

It is clear from this discussion that the binary counter nust have a capacity,

$$\begin{aligned} H &= \log_2 (a + \mid b \mid) \quad \text{bits} \\ &= \log_2 \left[\frac{\log \dfrac{1-\beta}{\alpha} + \log \dfrac{1-\alpha}{\beta}}{\log \dfrac{1-p_0}{1-p_1}} \right]. \end{aligned} \tag{19}$$

When $\alpha = \beta$, $a = |b|$, and we have

$$\begin{aligned} H &= 1 + \log_2 a \quad \text{bits} \\ &= 1 + \log_2 \left[\frac{\log \dfrac{1-\alpha}{\alpha}}{\log \dfrac{1-p_0}{1-p_1}} \right] \\ &\approx 1 + \log_2 \left[\frac{\log \dfrac{1}{\alpha}}{\log \dfrac{1-p_0}{1-p_1}} \right]; \qquad \alpha \ll 1. \end{aligned} \tag{20}$$

he capacity of the Bernoulli Detector is only a function f $(a, |b|)$. The behavior of the capacity as a function of he threshold parameters $(p_0, p_1, \alpha, \beta)$ is also indicated.

III. Measurement of the OC and ASN Functions

In that same paper[6] it is shown that the OC Function r the Bernoulli Detector is given by the parametric quations,

$$L_p(u) = \frac{u^a - 1}{u^a - u^b}; \qquad b < 0; \quad a > 0, \tag{21}$$

and

$$p(u) = \frac{u - 1}{u^{(c+1)} - 1} \qquad 0 \leq u \leq \infty. \tag{22}$$

When $u = 1$, we have

$$L_p(1) = \frac{a}{a + \mid b \mid} \tag{23}$$

$$p(1) = \frac{1}{c + 1}. \tag{24}$$

For the special case considered here, $a = |b|$, $(\alpha = \beta)$, we have

$$L_p(u) = \frac{u^a}{u^a + 1} \tag{25}$$

and

$$L_p(1) = \tfrac{1}{2}. \tag{26}$$

The ASN Function is given by

$$E_{p(u)}(n) = \frac{bL(u) + a[1 - L(u)]}{(1 + c)p(u) - 1}. \tag{27}$$

When $u = 1$, we have

$$E_{p(1)}(n) = \frac{a \mid b \mid}{c}. \tag{28}$$

For the special case $a = |b|$ $(\alpha = \beta)$, we have

$$E_{p(u)}(n) = \frac{a[1 - 2L(u)]}{(1 + c)p(u) - 1} \tag{29}$$

$$E_{p(1)}(n) = \frac{a^2}{c}. \tag{30}$$

The OC Function gives the probability of accepting the hypothesis H_0 when p is the parameter of the Bernoulli Distribution, while the ASN Function gives the average number of samples required for the termination of the Sequential Test when p is the parameter of the Bernoulli Distribution. Thus, from the experimental point of view, it is required to generate a family of Bernoulli random variables designated by the parameters, p, and to measure the OC and ASN Functions $L_p(u)$ and $E_{p(u)}(n)$, respectively.

Eqs. (21), (22) and (29) show theoretically that all of these functions are related to each other through the parameter u. The parameter is, however, not physically measurable directly and is of no interest except for analytical purposes. The quantities p, L_p, $E_p(n)$ are observable directly.

Let M be the total number of sequential experiments required to determine one point on the ASN Function (for example, for a given parameter p). Thus for a given set of constants (a, b, c) of the detector and for a given input random variable of parameter p, M decisions are made or equivalently a total of M crossings of the thresholds a and b are counted. Let $N(M)$ be the total number of samples (sum of zeros and ones) measured in the M sequential experiments. Thus the empirical ASN Function is given by

$$E_p(n, M) = N(M)/M. \tag{31}$$

The empirical ASN Function approaches the theoretical ASN Function with probability one, as M is allowed to increase indefinitely since the experiments are independent. Thus, for large M, the empirical ASN Function is for all practical purposes equal to the theoretical ASN Function.

We generate the Bernoulli Family of Distributions by sampling a band-limited low-pass white Gaussian noise source at sample points separated in time by an amount much greater than the reciprocal of twice the bandwidth. This insures us that the samples are, for all practical purposes, statistically independent. If σ is the parameter of the noise family, then a Bernoulli random variable can be generated by measuring the number of ones and zeros at the sample points at the output of a slicer which has the following nonlinear characteristic;

$$\begin{aligned} G[x_i(\sigma), x^0] &= 1 \qquad x_i(\sigma) \geq x^0 \\ &= 0 \qquad x_i(\sigma) < x^0. \end{aligned} \tag{32}$$

Thus for a given slicing threshold x^0, we can generate a family of Bernoulli Sequences by varying the parameter σ of the noise source. (Note that σ represents a labeling of the noise ensembles. It can be a dial control in a noise source or a gain control in an amplifier driven by a noise source or an ensemble of band-limited noise sources of different powers, etc.) It is not necessary to measure σ but only the number of zeros and ones at the output of the operator $G[x^0]$. Actually the only necessary *a priori* information is that the samples are statistically independent.

The frequency of ones in a total of M sequential experiments is given by

$$\nu_\sigma(M) = Y(N)/N(M) \tag{33}$$

where

$$Y(N) = \text{number of ones in } N(M) \text{ samples.}$$

For $N(M)$ large, it is known[8] that,

$$\text{Prob. } [\nu_\sigma(M) \to p] \to 1 \quad \text{as} \quad M \to \infty. \tag{34}$$

That is, the empirical probability is certain to converge to the mathematical probability as the number of samples is increased indefinitely. We can increase the total number of samples by increasing the number of Sequential Experiments and hence improve the approximation to the theoretical case. By increasing the number M we also improve the approximation to the ASN Function.

When the parameter $p_1 - p_0 \ll 1$, the ASN Function is sharply peaked. However, when this is the case, the number of samples required in the parameter range $p_0 \leq p \leq p_1$ for termination is large. Hence, we automatically gain precision in the measurement of the Bernoulli parameter p when it is most required. At the tails of the ASN Function which correspond to $p < p_0$, $p > p_1$, the total number of samples $N(M)$ is reduced, therefore giving less precision in the measurement of p. However, in this region the behavior of the ASN Function is not critical (and not very important from a practical viewpoint). We can therefore afford to lose some precision in p.

Another important point is that p and $E_p(n)$ are measured simultaneously. This not only results in a time saving but puts less restriction on the stability of the entire measuring apparatus. We require stability only over the measurement time interval corresponding to one point. This appears to us of great practical importance. The points on the ASN curve are not taken at equal intervals, but actually at random intervals.

The OC Function is also measured simultaneously with the ASN Function. This function gives the probability of accepting the hypothesis H_0 when p is the Bernoulli parameter. Thus to measure this function, it is necessary to count the number of times in M sequential experiments that the reversible binary counter (discussed in Section II) returns to zero. Let $L_p(M)$ be this number. Then

$$L_p(M) = \frac{\alpha(M)}{M} \tag{35}$$

where $\alpha(M)$ is the number of times in M experiments that the hypothesis H_0 is accepted. Of course, $M - \alpha(M)$ is the number of times that H_1 is accepted. Since the experiments are independent, we know that the empirical OC Function of (35) is certain to approach the mathematical OC Function at each point as the number of sequential experiments is allowed to increase indefinitely.

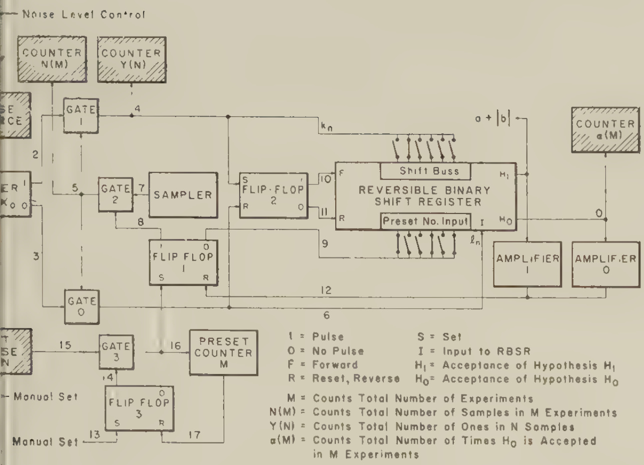
For each parameter σ of the noise source, we measure the Bernoulli parameter p and the corresponding point on the OC and ASN Functions simultaneously. We emphasize again that this approach increases the speed of measurement and reduces the requirements on the stability of the operations. The variable precision in p automatically adjusts itself so that more precision is obtained when required and less precision when not required. For example at the indeterminate point $p' = 1/(c + 1)$, which is an important point in the measurement, the ASN Function is a maximum, hence the total number of samples $N(M)$ measured for the estimate of p' is also a maximum. We emphasize this point as it is well-known that the functional behavior of a sequential test is the same for a very general class of one parameter family of distributions that are of practical interest. Thus the type of measurements described here can also be made on other statistical variables with the same advantages as for Bernoulli statistics.

IV. Functional Block Diagram of a Sequential Bernoulli Detector and Discussion of Experimental System Operation

Fig. 1 is a block diagram of the Bernoulli Sequential Detector and the experimental apparatus necessary to obtain the OC and ASN Functions. A detailed description of the logical operation of the system will now be presented.

The primary source for generating the Bernoulli Sequence is a noise source whose output is a Gaussian signal having a flat power spectrum. The output at

[8] M. Loeve, "Probability Theory," D. Van Nostrand Co., Inc., New York, N. Y.; 1955.



1—Sequential Bernoulli filter and experimental apparatus.

ninal 1 is fed into a slicer whose dynamic characstic is given by (32). That is, the slicer output is a dom square wave of unit height corresponding to the e intervals during which $f(t)$ is above the threshold. ice, at terminal 2 the positive pulses correspond to ones in the sequence $(x_i \geq x_0)$ and at terminal 3, positive pulses correspond to the zeros in the sequence $< x_0)$. The sequence of zeros and ones is fed into gates and G_1 respectively which are normally cutoff.

he sampler output at 7 is simply a series of narrow es which occur at a constant repetition period T. repetition period is $T \gg 1/2W$ where W is the bandth of the noise. When G_2 is gated by FF_1, the sampling es at 7 are gated through and appear at 5. These are lied to G_0 and G_1. The output of G_1 is a sequence of es appearing at the sampling points which correspond the values for which $x_i \geq x_0$. The output at G_0 is a uence of pulses appearing at the sampling points which respond to the values $x_i < x_0$. Hence, the pulses at 4 6 correspond to the sequence of ones and zeros pectively. The output at 4 is applied to terminal S F_2. Each time a one is observed, FF_2 is triggered into e one, in turn selecting the forward-count direction of reversible binary shift register, RBSR. On the other d, when a zero is observed, a pulse applied to R of selects the reverse-count direction of the RBSR. en terminal F of RBSR is selected, a pulse correspond- to a one is applied to the shift bus. This shifts in the nber c into the RBSR. When R is selected, a pulse responding to a zero is applied to the input terminal I. s shifts in the number one into the RBSR (corresnding to an observed zero). During this time the SR remains preset to the number $|b|$. When the SR returns to zero this is sensed at H_0, indicating the eptance of H_0 that noise is present. If it moves up to count $a + |b|$, this situation is sensed at H_1 indicating t signal-plus-noise is present. At these instants a pulse pplied either to amplifier A_0 or A_1 which appears at and resets FF_1. This in turn gates out G_2 and hence and G_0, essentially disconnecting all input information the RBSR. The system remains inactive until the external pulse generator EPG applies a trigger to G_3. At this instant FF_1 is set gating on G_2. At the same time the gate at 9 is applied to the preset terminal P. This presets the RBSR to the count $|b|$. The system continues operating until a pulse is generated at either H_0 or H_1 indicating that a decision has been made. The pulse repetition period of the EPG is adjusted to be sufficiently greater than the time it takes to come to a decision. If necessary, the system can be started by triggering the EPG manually. It is also possible to start a new experiment by deriving a trigger from the decision output.

The number of sequential experiments is equal to the number of pulses generated by the EPG. This number is counted by the preset counter M, that is, the manual set at 13 triggers FF_3 gating on G_3. This permits pulses from the EPG to be gated through G_3, thus beginning a sequential experiment. When M pulses indicating M sequential experiments have entered the preset counter, FF_3 is automatically reset at 17, stopping the entire system.

The M counter reads the total number of sequential experiments. The $\alpha(M)$ counter reads the total number of times in M experiments that H_0 was accepted or that the reversible counter returned to zero. The N counter reads the total number of samples in M experiments while the $Y(N)$ counter reads the total number of ones in M experiments. After M experiments this information is read by the observer from the counters. The information is sufficient to give a single point on the ASN and OC Functions directly.

V. Choice of Detector Constants and Number of Experiments Per Point

The total number of sequential experiments per point was chosen as $M = 100$. The values of the detector constants chosen for the experimental study of the theory are

$$\begin{aligned} a &= -b = 16, \\ a &= -b = 32, \\ c &= 4, \\ c &= 8. \end{aligned} \tag{36}$$

This defines four different physical experiments, for example,

$$\begin{aligned} a = 16; \quad & c = 4 \\ a = 32; \quad & c = 4 \\ a = 16; \quad & c = 8 \\ a = 32; \quad & c = 8. \end{aligned} \tag{37}$$

Keeping $c = 4$ and allowing $a = 16$ and $a = 32$ is equivalent to specifying an allowable set of values (p_0, p_1) and varying the set $\alpha_1 = \beta_1$ to $\alpha_2 = \beta_2$. From Fig. 2 corresponding to $c = 4$ an allowable set of values p_0, p_1 is

$$\begin{aligned} p_0 &= 0.165, \\ p_1 &= 0.240. \end{aligned} \tag{38}$$

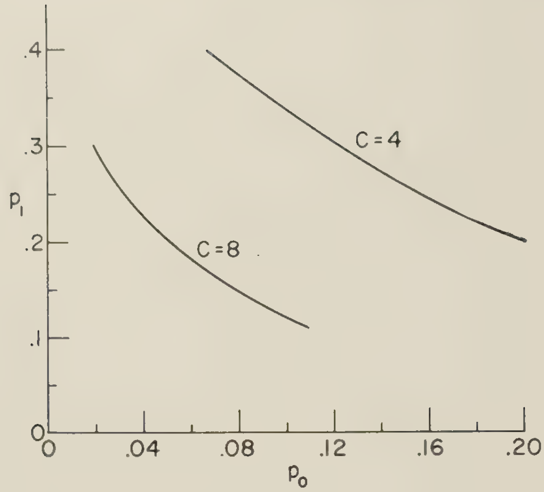


Fig. 2—Graph of p_1 vs p_0 for two values of c.

Let,

$$a = 16.$$

The function

$$\log \frac{1 - p_0}{1 - p_1} = 0.0942. \tag{39}$$

From (15)

$$\frac{\log \dfrac{1-\alpha}{\alpha}}{\log \dfrac{1 - p_0}{1 - p_1}} = 16.$$

Hence,

$$\alpha = \beta = 0.18. \tag{40}$$

For the same set of values (p_0, p_1) and $a = 32$

$$\alpha = \beta = 0.0475. \tag{41}$$

From Fig. (2) when $c = 8$, an allowable state is given by

$$\begin{aligned} p_0 &= 0.09, \\ p_1 &= 0.135. \end{aligned} \tag{42}$$

Hence,

$$\log \frac{1 - p_0}{1 - p_1} = 0.0507. \tag{43}$$

Similar to the previous example, when $a = 16$

$$\alpha = \beta = 0.3. \tag{44}$$

When $a = 32$,

$$\alpha = \beta = 0.156. \tag{45}$$

It should be clear that if another allowable pair of values (p_0, p_1) were taken, the corresponding α and β would be different.

Because the Wald Theory is only approximate for discrete sampling, since the excess over the boundaries at the termination of experiment is neglected, it is desirable to obtain upper and lower bounds on the OC and ASN Functions. These are derived in Wald.[7] It can be shown that for the constants used, the lower bounds are for all practical purposes those given by (25) and (29). For the constants the true lower bounds are only slightly low (a negligible amount for practical purposes). The upp bound for the OC Function is given by Wald,[7]

$$L'(u) = \frac{u^{2a \cdot c} - u^{a}}{u^{2a+c} - 1}, \quad (4$$

$$L'(1) = \frac{a + c}{2a + c}. \quad (4$$

An upper bound on the ASN Function is also given,[7]

$$E'_u(n) = \frac{a[1 - 2L(u)]}{(1 + c)p(u) - 1} + \frac{c[1 - L(u)]}{(1 + c)p(u) - 1}. \quad (4$$

Eq. (48) is well-behaved everywhere except in the neig borhood of $u = 1$. In this region we have the relationsh given in (30).

VI. Discussion of Experimental Results in the Light of the Theory

Before attempting to correlate experiment and theor some of the unobservable theoretical characteristics w be discussed. Fig. 2 is a plot of the allowable paramet points corresponding to the parameter values $c = 4$ ar $c = 8$. It is clear that there exists an infinite number such parameter points. This is consistent with the previo discussion. The function $c(p_0, p_1)$ is plotted in Fig. 3. F a given value of c, the corresponding set of pairs of valu (p_0, p_1) can be obtained from this family of functions.

Fig. 4 is a plot of (22), and Fig. 5 of (25). The importa thing to notice is that the functions are monotomic in tl strict sense.

Fig. 6 is a theoretical plot of the upper and lower boun on the OC Function corresponding to $a = |b| = 3$ $c = 4$. The points represent experimental data. The upp and lower bounds almost overlap each other. Wa predicts for the case where the mean and standard devi tion of the measured observable is small, neglecting tl excess over the boundaries would have little affect on tl OC and ASN Functions. Hence, the experimental poin should follow the lower bound. Thus Wald's work show that the parameter c must be small in order for the effe of neglecting the excess over the boundaries to be sma For $p_0 = 0.08$, $p_1 = 0.375$, corresponding to $c = 4$ tl experimental points follow what is essentially the low bound. For this case, the lower and upper bounds a essentially the same. The small deviations of the cur are statistical fluctuations since only 100 experiments a performed. This is evident by the manner in which th are distributed. Most of the deviations occur in tl high slope region for example, in the neighborhood p $1/(c + 1)$. This is the region which requires a limit pr cess for convergence.

Fig. 7 is a curve of the corresponding ASN Functio Once again the experimental points follow the low bound. Note the very close correlation of theory ar experiment to the left of the maximum. This is due the fact that for $p < 0.18$, the hypothesis H_0 is accept

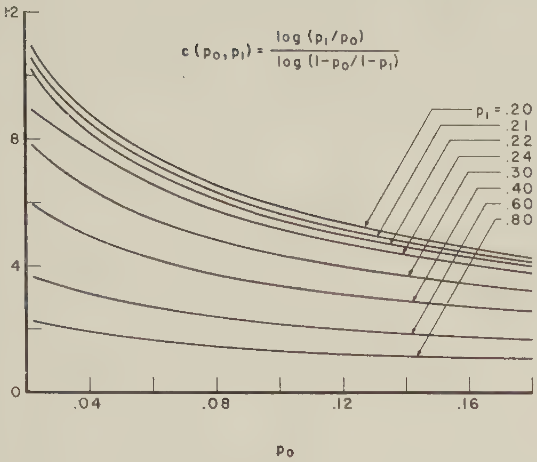
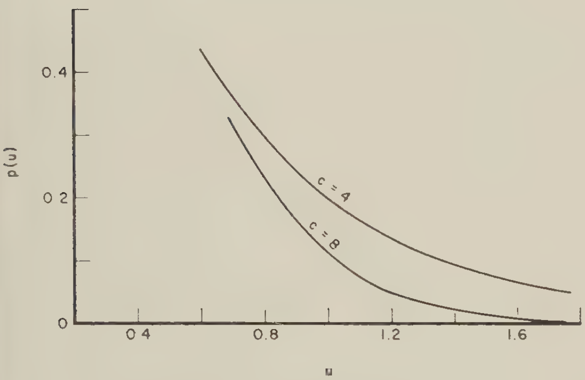
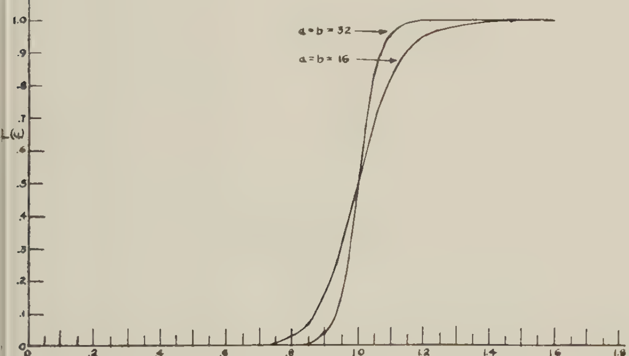


Fig. 3—Family of curves of $c(p_0, p_1)$.



. 4—Universal characteristic of Bernoulli sequential filter in region of physical interest.



. 5—Universal operating characteristic function of sequential filter.

the average approximately 85 per cent of the time. nce, the reversible shift register returns to zero 85 cent of the time. When H_0 is accepted there is no ess over the boundaries at the termination of the periment. That is, the equality sign is realized. There-e, in this interval, theory and experiment agree very sely. On the right side of the peak, the experimental nts are very close to the lower bound. However, there sts a consistent discrepancy between theory and periment. It is possible that during this phase of the instead of the test terminating with H_1 corresponding a value $a = 32$, it might have terminated at times

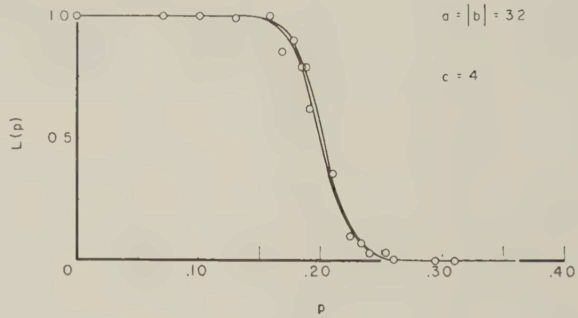


Fig. 6—Upper and lower bounds on the OC function of Bernoulli-sequential filter and the experimental points.

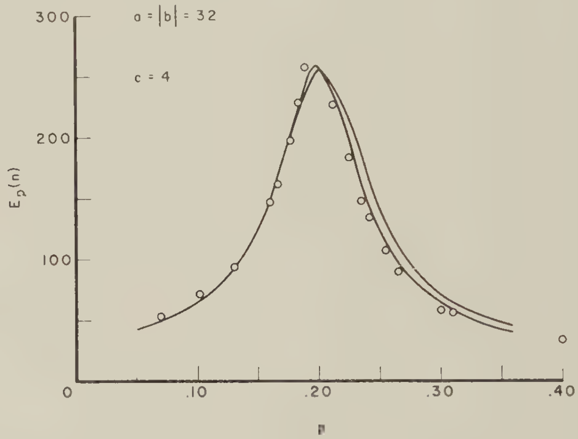


Fig. 7—Upper and lower bounds on the ASN function of Bernoulli-sequential filter and the experimental points.

with $a = 30$ or some number less than 32 although close to it. To detect such an error is almost a physical impossibility since it occurs when the system is operating in its most complex manner. That is, it is unlikely that a signal of constant repetition period would isolate an error of this nature. The error although systematic is still too small to be of practical consequence. It is also possible that the lower bound is not sufficiently good. It gives pessimistic results or results that are always on the safe side. This situation is always desirable.

The points were obtained simultaneously by reading the four counters. It is clear from the ASN Function that the maximum number of samples taken for a point is at the peak or approximately 26,000 samples. This is the most critical region. This gives a precision in p of the order of 0.6 per cent. The precision in $L(p)$ and $E_p(n)$ is of the order of 10 per cent. The statistical characteristics of these functions are apparently of such a nature that such a precision is sufficiently good. This is indicated by the results. It is felt that the simultaneous measurement of these characteristics is the reason for good agreement with not too large a sample. Fig. 8 is a plot of the OC Function for the same values of c but for $a = |b| = 16$. It is clear once again that the experimental points follow the lower bound. This indicates that for small values of c, Wald's Theory holds equally well for discrete sampling. The parameters (a, b) apparently have no influence on the characteristics, as predicted by Wald. The trend

of the points indicates that the lower bound may be slightly high. The difference is however, insignificant.

The corresponding ASN Function, Fig. 9, also shows that the experimental points follow the lower bound. The fluctuation in the points is of a statistical nature.

By comparing Figs. 6 and 8 it is seen that the effect of increasing the numbers (a, b) is to increase the slope of the OC Function. This corresponds to a decrease in the parameters (α, β).

A comparison of Figs. 7 and 9 shows that an increase in (a, b) peaks up the ASN Function in the neighborhood of $p = 1/(c + 1) = 0.20$. From these curves it is also seen that the ASN Function is a maximum at approximately $p = 1/(c + 1) = 0.20$ as predicted by theory. This is the slope of the statistical decision lines. The theory does not say that the maximum occurs precisely at $p = 1/(c + 1)$. It is very close, hence for all practical purposes, it can be considered as the point at which the ASN Function is a maximum.

Fig. 10 is a plot of the OC Function for $a = b = 32$, the same as Fig. 6, and for $c = 8$. It is seen that for $c = 8$, the OC Function follows the upper bound. It appears, however, that there is a systematic trend for the points to be above the upper bound, especially in the region $0.10 < p < 0.17$. The experimental curve is shifted almost parallel to the lower bound. This is contrary to the way the upper bound behaves. There is no obvious reason for experiment and theory to differ in this manner. Intuitively we might expect the upper and lower bounds to be parallel in the almost linear region as indicated by experiment. A look into this matter strictly from a theoretical point of view might be desirable. From the practical point of view, the agreement between theory and experiment is sufficiently good.

Fig. 11 is the corresponding ASN Function. Once again it is noted that the curve to the left of the maximum agrees with Wald's Theory, while the part to the right follows the upper bound. Most of the points on the right are less or equal to the upper bound. The maximum of the ASN Function occurs at the point given by theory $p = 1/(c + 1) = 0.11$.

Fig. 12 is a plot of the OC Function for $a = |b| = 16$ and $c = 8$. Most of the points lie within the two bounds. Once again the trend of the experimental points in the almost linear region is to lie on a curve parallel to the lower bound. This agrees with Fig. 10. It should be emphasized that this is probably inherent in the approximation of the upper bounds. Fig. 13 is the ASN Function corresponding to $a = |b| = 16$, $c = 8$. Once again the points follow the upper bound to the right of the maximum. To the left of the maximum, the experimental points exceed the theoretical bounds. The maximum of this curve is very broad. Hence, the neighborhood of the limit point $p = 1/(c + 1) = 0.11$ is large. This implies that the Wald bounds are not very good over a wider interval. The points corresponding to $p > 0.15$ do follow the upper bound. The validity of the approximation is therefore shown to be good in this region. It should be pointed out

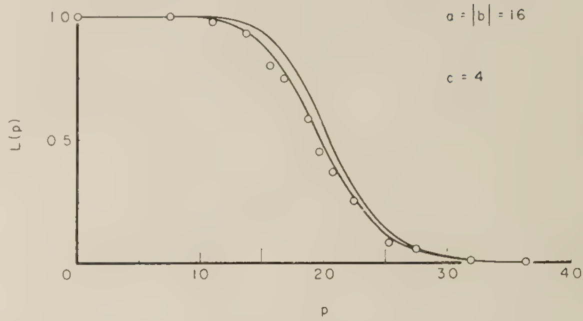


Fig. 8—Upper and lower bounds on the OC function of Bernoul sequential filter and the experimental points.

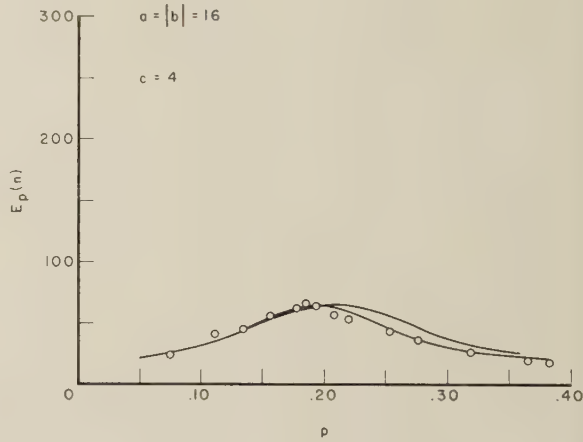


Fig. 9—Upper and lower bounds on the ASN function of Bernoul sequential filter and the experimental points.

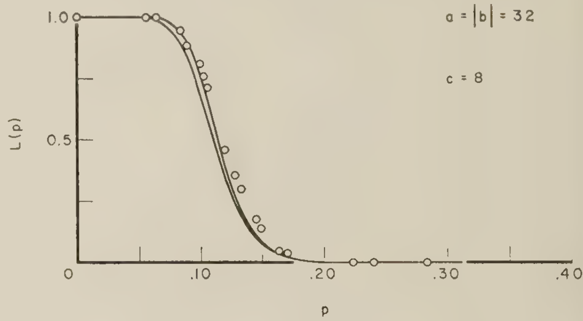


Fig. 10—Upper and lower bounds on the OC function of Bernoul sequential filter and the experimental points.

that a plot of the upper bound in the nieghborhood $p = 1/(c + 1)$ will produce a violent oscillation. This not shown since it simply was an indication that t bounds are not good in that region. The curve drawn that interval is somewhat of an extrapolation.

VII. Conclusions

In general, the agreement between the Wald Theo which neglects the excess over the boundaries at t termination of a sequential experiment and the results f discrete sampling is good. The upper and lower boun on the OC and ASN Functions are also good for practic purposes. The discrepancies in general are not of a stati tical nature. They are small but systematic and are probab due to the fact that the expressions for the bounds al

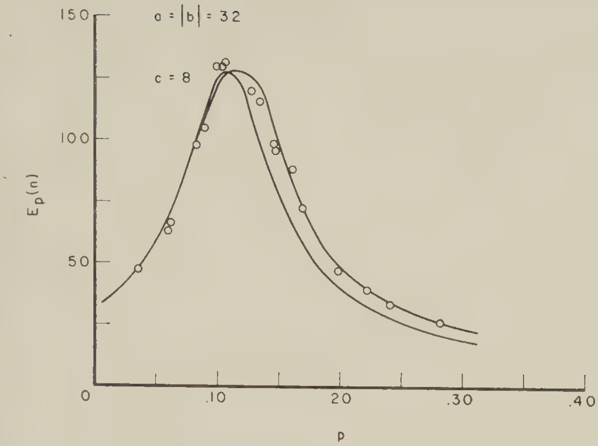


Fig. 11—Upper and lower bounds on the ASN function of Bernoulli-sequential filter and the experimental points.

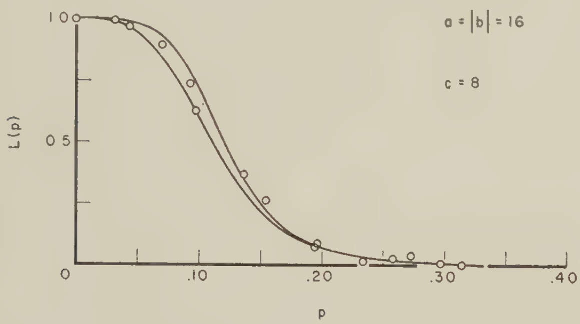


Fig. 12—Upper and lower bounds on the OC function of Bernoulli-sequential filter and the experimental points.

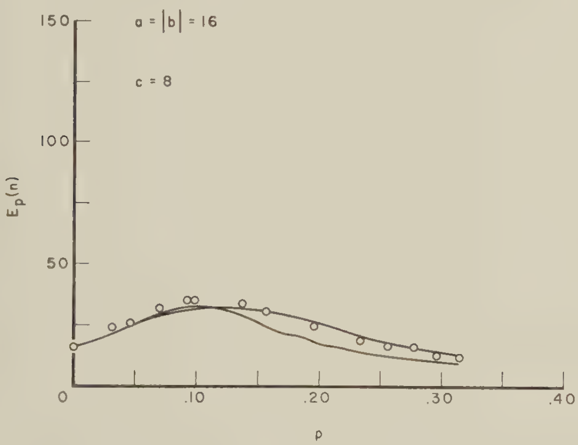


Fig. 13—Upper and lower bounds on the ASN function of Bernoulli-sequential filter and the experimental points.

involve certain approximations which are claimed to have only a small effect on the results. The experiments bring this out.

The experimental results verify the transformed equations in terms of the u variable. It is shown conclusively that only the (a, b, c) transformations are significant in the sequential observation. This is important since it shows which parameters are of practical significance and which are only meaningful conceptually. It shows that the approximations depend on a functional relationship of $(p_0, p_1, \alpha, \beta)$ and not directly on these parameters. The experimental results actually are true for an infinity of Bernoulli Sequential experiments which satisfy the given (a, b, c) transformations.

VIII. Addendum

The Resonant Properties of a Sequential Detector

One of the most striking phenomena observed during the experiments was the behavior of the ASN Function in the neighborhood of the indeterminate point $p' = 1/(c + 1)$. As seen from the curves, in this region a resonance phenomenon is observed. If we use the random walk interpretation of sequential detection, then the concept of resonance in sequential analysis is clear. For example, sequential detection can be considered as a random walk problem with two absorbtion barriers. Thus the particle is trapped between the two barriers (a, b) with a probability unity that it will emerge eventually. There is, however, one distribution corresponding to a given set of threshold constants (a, b, c) which will force the particle to oscillate, at random of course, back and forth about its zero position (in this case about the preset number $|b|$) and hence spend the maximum time, on the average, between the absorption barriers. This is the point at which the ASN Function is determined by a limit process. The sharpness of this resonance can, of course, be made as closely as desired by choosing the parameters (p_0, p_1) or in general (θ_0, θ_1) as close to each other as desired. The sharpness of this resonance has a very striking similarity to the usual resonance phenomenon encountered in a linear RLC circuit. If we stretch our concepts somewhat, we can say that a sequential detector of given constants (a', b', c') is a "statistical resonant circuit tuned to the distribution parameter $\theta = \theta'$." Let us therefore examine the properties of the sequential detector by introducing the concepts inherent in resonant circuit theory. This will give further insight into the behavior of sequential detectors.

Let us consider the familiar one parameter family of distributions given by

$$P(x, \theta) = v(\theta)w(x)e^{xq(\theta)} \tag{49}$$

which is a special case of the family of distributions given in (1). This class includes the familiar and important distributions such as the one parameter Gauss, the Bernoulli, Poisson, etc. For this special case it can be shown that the decision regions are given by

$$\sum_{i=1}^{n} x_i \geq a' + nc' \rightarrow H_1 \tag{50}$$

$$\sum_{i=1}^{n} x_i \leq b' + nc' \rightarrow H_0, \tag{51}$$

where

$$a' = \frac{\log \dfrac{1-\beta}{\alpha}}{q(\theta_1) - q(\theta_0)}, \tag{52}$$

$$b' = \frac{\log \frac{\beta}{1-\alpha}}{q(\theta_1) - q(\theta_0)}, \tag{53}$$

$$c' = \frac{\log \frac{v(\theta_0)}{v(\theta_1)}}{q(\theta_1) - q(\theta_0)}. \tag{54}$$

The ASN Function, which is the characteristic of interest for this discussion, is given by

$$E_\theta(n) = \frac{b'L(\theta) + a'[1 - L(\theta)]}{E_\theta(x - c')}. \tag{55}$$

At the indeterminate point $\theta = \theta'$ corresponding to the value of θ for which $E_\theta(x) = c'$, we have for the ASN Function,

$$E_{\theta'}(n) = -\frac{a'b'}{E_{\theta'}(x - c')^2}. \tag{56}$$

For the Bernoulli case $E_{\theta'}(x - c')^2 = p'(1 - p')$ and with the relationships between (a', b', c') and (a, b, c) one can obtain (30).

It is known that the ASN Function is for all practical purposes a maximum at $\theta = \theta'$. At the threshold parameters $\theta = \theta_0$; and $\theta = \theta_1$, $L(\theta_0) = 1 - \alpha$; $L(\theta_1) = \beta$. Hence,

$$E_{\theta_0}(n) = \frac{\alpha(a' - b') + b'}{E_{\theta_0}(x - c')} \tag{57}$$

and

$$E_{\theta_1}(n) = \frac{-\beta(a' - b') + a'}{E_{\theta_1}(x - c')}, \tag{58}$$

knowing that $E_{\theta_0}(n) < E_{\theta'}(n)$ and $E_{\theta_1}(n) < E_{\theta'}(n)$. From previous considerations we have,

$$E_\theta(x - c') = E_\theta(x) - E_{\theta'}(x) \tag{59}$$

and

$$E_{\theta'}(x - c')^2 = E_{\theta'}\{[x - E_{\theta'}(x)]^2\} = \sigma_{\theta'}^2(x). \tag{60}$$

Thus the maximum of the ASN Function is given by (56) which when combined with (60) gives

$$E_{\theta'}(n) = -\frac{a'b'}{\sigma_{\theta'}^2(x)}. \tag{61}$$

We now define the value of the ASN Function at the threshold parameters (θ_0, θ_1) as,

$$E_{\theta_0}(n) = E_{\theta_1}(n) = kE_{\theta'}(n): \qquad k < 1. \tag{62}$$

Substituting (57) and (58) into (62) and using equation (59), then solving for $E_{\theta_0}(x)$ and $E_{\theta_1}(x)$ yields

$$E_{\theta_0}(x) = E_{\theta'}(x) - \frac{\alpha(a' - b') + b'}{ka'b'}\sigma_{\theta'}^2(x) \tag{63}$$

and

$$E_{\theta_1}(x) = E_{\theta'}(x) - \frac{-\beta(a' - b') + a'}{ka'b'}\sigma_{\theta'}^2(x). \tag{64}$$

For the class of distributions considered, the functions $E_\theta(x)$ are monotonic increasing with θ in the strict sense. We can now define a bandwidth,

$$W = E_{\theta_1}(x) - E_{\theta_0}(x)$$
$$= -\frac{1 - (\alpha + \beta)}{ka'b'}(a' - b')\sigma_{\theta'}^2(x). \tag{65}$$

Since $E_\theta(x)$ is monotonic in θ, we know that for a given bandwidth $w = \theta_1 - \theta_0$ there corresponds a bandwidth $W = E_{\theta_1}(x) - E_{\theta_0}(x)$. We can therefore specify the bandwidth w by specifying θ_1 and θ_0 and this in turn determines W which is required in (65). (In certain cases such as the Bernoulli case $w = W = p_1 - p_0$.)

For convenience, let $E_{\theta'}(n) = N$. Then solving (61) for b' and substituting into (65) gives the quadratic equation

$$(a')^2 - a'\frac{WN}{\left[\frac{1 - (\alpha + \beta)}{k}\right]} + N\sigma_{\theta'}^2(x) = 0. \tag{66}$$

When (66) is solved for a' we have

$$a' = \frac{WN}{2\left[\frac{1 - (\alpha + \beta)}{k}\right]} \pm \frac{1}{2}\sqrt{\left[\frac{WN}{\frac{1 - (\alpha + \beta)}{k}}\right]^2 - 4N\sigma_{\theta'}^2(x)}. \tag{67}$$

In order for (67) to make sense, it is necessary that

$$\left[\frac{WN}{\left(\frac{1 - (\alpha + \beta)}{k}\right)}\right]^2 \geq 4N\sigma_{\theta'}^2(x)$$

or

$$\left(\frac{W}{2}\right)\left[\frac{k}{1 - (\alpha + \beta)}\right] \geq \frac{\sigma_{\theta'}(x)}{N}. \tag{68}$$

We also know that $a' > 0$. The right part of (68) is recognized as a familiar relationship in normal statistics. It can be easily verified that for the case

$$\left(\frac{k}{1 - (\alpha + \beta)}\right)\frac{W}{2} = \frac{\sigma_{\theta'}(x)}{\sqrt{N}}, \tag{69}$$

we have, $a' = -b'$ or $\alpha = \beta$. Furthermore,

$$a' = -b' = \frac{k}{1 - 2\alpha}\left(\frac{WN}{2}\right) = \sqrt{N}\,\sigma_{\theta'}(x). \tag{70}$$

For this special case, the decision regions are given by

$$\frac{1}{N}\sum_{i=1}^{n}[x_i - E_{\theta'}(x)] \quad \begin{matrix} \geq \frac{W}{2}\left(\frac{k}{1 - 2\alpha}\right) \to H_1 \\ \leq \frac{W}{2}\left(\frac{k}{1 - 2\alpha}\right) \to H_0. \end{matrix} \tag{71}$$

It should be clear that once W is specified in terms of θ_0 and θ_1, $E_{\theta'}(x)$ is also specified since $E_{\theta'}(x)$ is a function of θ_1 and θ_0. Since interest here lies in the resonance phe-

[phen]omenon, let us rewrite (71) as,

$$-\frac{k}{1-2\alpha}\frac{W}{2} \le \frac{1}{N}\sum_{i=1}^{n}[x_i - E_{\theta'}(x)] \le \frac{W}{2}\frac{k}{1-2\alpha}. \quad (72)$$

Let us assume that it is desired to determine if a random variable belongs to a distribution whose parameter is θ' where $\theta_0 \le \theta' \le \theta_1$. When the unknown parameter is $\theta = \theta'$ it is known that the number of samples N required for the sequential test to terminate is a maximum. If $(\theta_1 - \theta_0)$ is very small, there is a sharp maximum. We therefore specify a number N_0 such that, if $N \ge N_0$, then it is known with a predetermined probability that $\theta_0 \le \theta \le \theta_1$, and if $N < N_0$, then $\theta < \theta_0$, or $\theta > \theta_1$. We can therefore consider the sequential filter when operating in this mode as a parameter filter which operates in a manner which is clearly analogous to a resonant circuit. For example, if the input to a resonant circuit is one of a set of sinusoidal signals and if it is required to determine if a particular frequency is present, then the resonator is tuned to this frequency, and the response is measured to determine the presence or absence of a sinusoid at that frequency. Similarly in the Sequential Filter described, "tuning" is accomplished by adjusting the value of $E_{\theta'}(x)$ for a particular θ'. If the "response" is $n \ge N_0$, then a random variable whose distribution has the parameter θ is detected. In this same way, we can use a bank of sequential detectors "tuned" over an entire range of parameters and recognize or estimate one of a set of predetermined parameters. It therefore appears that in this mode of operation the sequential detectors can be used in multivalued decision problems. The decision to accept a hypothesis is made by comparing the responses (number of samples) at the output of each sequential detector after experimentation has terminated and choosing the hypothesis which corresponds to the largest response (maximum number of samples). It should be clear that the random variables need not belong to one family but can belong to any member of the exponential class.

In order to illustrate these ideas more clearly, we will consider (72) in more detail. From (70), we can rewrite (72) as,

$$-1 \le \sum_{i=1}^{n}\frac{[x_i - E_{\theta'}(x)]}{\sqrt{N}\,\sigma_{\theta'}(x)} \le 1. \quad (73)$$

Let $n = kN$ be the minimum number of samples required to accept the hypothesis that $\theta_0 \le \theta \le \theta_1$, that is, that the parameter θ is contained in the confidence interval (θ_0, θ_1). We can use this fact in (73), which gives

$$-\frac{1}{\sqrt{k}} \le \sum_{i=1}^{kN}\frac{[x_i - E_{\theta'}(x)]}{\sqrt{kN}\,\sigma_{\theta'}(x)} \le \frac{1}{\sqrt{k}}. \quad (74)$$

For $kN \gg 1$ (of the order of 100), the random variable,

$$Y = \frac{\sum_{i=1}^{kN}[x_i - E_{\theta'}(x)]}{\sqrt{kN}\,\sigma_{\theta'}(x)}, \quad (75)$$

is approximately normal with mean zero and standard deviation unity. Let

$$\Gamma_{\theta'} = P_{\theta'}\left\{-\frac{1}{\sqrt{k}} \le Y_{\theta'} \le \frac{1}{\sqrt{k}}\right\} \quad (76)$$

be the probability that when θ' is the true parameter, the random variable Y will be observed in the designated interval. Thus, the constant k previously defined as

$$k = \frac{E_{\theta_0}(n)}{N} = \frac{E_{\theta_1}(n)}{N},$$

is all that is required to find the probability corresponding to a given confidence interval when the number of samples is large, for the entire class of distributions considered here. It can be shown that corresponding to $\Gamma = 0.95$ we have $k = 0.25$. Thus in designing such a statistical resonant filter, we specify the threshold parameters (θ_0, θ_1). This determines $E_{\theta'}(x)$ and $\sigma_{\theta'}(x)$ or W. The constant k which defines the confidence interval is then chosen for a given confidence probability. Furthermore, kN is chosen sufficiently large so that the large sample distribution of the random variable Y is approximately normal. This, of course, defines the peak response N and the entire sequential filter. In this mode of operation the probability $\Gamma_{\theta'}$ is used instead of α, although a specification of either determines the other.

The indicated theory can be used to design a probability distribution analyzer which measures the empirical distribution of an independent set of samples. This can be accomplished by defining a set of slicing thresholds on the random signal such that the output of the ith slicer is,

$$G(x_i) = u_i = 1; \qquad x \ge x_i$$
$$= 0 \text{ otherwise.} \quad (77)$$

The slicer output is a set of zeros and ones distributed in a Bernoulli Distribution of probabilities $(p_i, 1 - p_i)$. It is required to estimate the $p(x_i)$ for each slicing threshold. For the Bernoulli case,

$$E_{\theta'}(x) = p', \qquad \sigma_{\theta'}(x) = \sqrt{p'(1-p')}. \quad (78)$$

For a typical point p' on the distribution and for a 95 per cent confidence interval, (74) becomes

$$-1 \le \sum_{i=1}^{.25N}\frac{[u_i - p']}{\sqrt{N}\,p'(1-p')} \le 1. \quad (79)$$

In order to find the value of p' we "tune" the filter by varying p' until the "response" is $n \ge kN$. When this occurs, we have found the required value of probability corresponding to the particular slicing threshold. The standard way of performing this measurement is to count the total number of samples and the number of ones. The frequency at the threshold is obtained by taking the ratio of ones to total number of samples. The latter method can be considered as analogous to measuring the frequency of a sinusoid by counting rather than measuring the response of a resonant circuit.

Minimum-Scan Pattern Recognition*

ARTHUR GILL†

Summary—Speedier and simpler pattern-recognition systems can be realized when provided with a minimum-scan reading device. For the idealized case, where the input set of patterns is finite, binary and errorless, a theorem is proved which enables the designer to predict the efficiency range of the contemplated reading device. A constructive method, which can be readily programmed for computer processing, is proposed for finding the shortest scanning path realizable for any given set. In the case of noise, scanning paths are sought which maintain a prescribed minimal "distance" between the patterns, and hence yield a prescribed level of error-detecting capability. The theorem previously proved is extended for this case, and a constructive method is proposed for finding the shortest path consistent with any specified minimal distance, for any given set of patterns.

Introduction

THE problem of automatic processing of printed documents has been receiving an increasing amount of study in recent years. Various character-reading systems have been designed, and a number of papers have been published with special emphasis on the mechanical and electronic aspects of the pattern-scanning mechanism, or on the judicious coding of the read-in material. In virtually all the proposed systems the *reader*—the device which transfers the given pattern information to the coder—is designed to scan each pattern completely, without any regard to the redundancies which may be introduced. The entire task of removing the redundancies is imposed on the coder which, following some statistical criteria, converts the scanned information into code-words suitable for fast and reliable recognition.

In this paper, the problem of designing a more "intelligent" reader is considered. Such a reader will not scan the entire area of each given pattern, but a judiciously-chosen portion of it. The result in many practical cases is a considerably speedier operation, as well as simplification in the coding and recognizing parts of the system. The scanned portion of each pattern—or the scanning path—is designed to be the same for all possible input characters; consequently the reader is required to have a fixed mechanism to make it ignore a predetermined portion of the pattern area. Thus the higher speed and simplification may, in terms of hardware, be achieved quite inexpensively.

The paper is restricted to preliminary analysis where the following assumptions are made: 1) the number p of different patterns which may be fed into the recognizing system is finite; 2) all patterns can be divided into a finite number of cells c, each of which is completely black or completely white; 3) a perfect positioning mechanism is available.

Under these assumptions, the set of different input patterns can be conveniently represented by a $p \times c$ matrix of binary digits, where unities and zeros represent black and white cells respectively. Thus, in such a matrix, the element common to row i and column j will describe the j'th cell of the i'th pattern. Matrix (1) below is an example of a pattern matrix with $p = 6$ and $c = 9$; the indexes attached to the rows and columns are the serial numbers of the patterns and cells respectively.

A Pattern Matrix

(1)

$$\begin{array}{c c} & 123456789 \\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix}111001111\\111001001\\110100101\\101001101\\001110100\\000101110\end{bmatrix}\end{array}$$

Pattern Matrix Reduction

For the idealized input described above and under the assumption of errorless reading, the most efficient reader would be the one which can distinguish between the various patterns while scanning the minimum number of cells. Thus, our first task in designing the reader is to find the minimal scanning path realizable for the given set of patterns. In terms of the pattern matrix, this task corresponds to finding the largest number of columns which can be deleted so that the remaining rows will still be distinct. The result of this reduction operation is a minimal pattern matrix which, out of all irreducible pattern matrices, has the smallest number of columns. Before describing any minimal reduction schemes, it is useful to state and prove the following theorem:

If $p \times c$ is the dimension of the original pattern matrix and $p \times c_{\min}$ the dimension of the corresponding minimal matrix, then:

$$\{\log_2 p\} \leq c_{\min} \leq p - 1$$

where the notation $\{x\}$ represents the smallest integer larger than or equal to x.

Proof

The total number of different patterns constructable from $c_{\min}$ cells is $2^{c_{\min}}$; hence:

$$p \leq 2^{c_{\min}}$$

or:

$$c_{\min} \geq \log_2 p.$$

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nce $c_{\min}$ has to be an integer, the last inequality is uivalent to:

$$c_{\min} \geq \{\log_2 p\}$$

hich verifies the lower limit in the theorem.

To verify the upper limit, consider the original $p \times c$ atrix in which the rows have been arranged in a sequence decreasing numerical value (each row being regarded a binary number). Now proceed from left to right and ark the column in which row 1 and row 2 first disagree. epeat with rows 2 and 3, 3 and 4, down to $p - 1$ and Deleting all unmarked columns leaves a matrix which as no more than $p - 1$ columns. To show that in the duced matrix no two rows are identical, consider any air of consecutive rows, say i and $i + 1$, and suppose at they first disagree in column j. These rows are then entical to the left of j; in column j row i is unity and w $i + 1$ is zero. Consequently, in the reduced matrix well as in the original matrix, row i has to be numerically rger than row $i + 1$. Since this argument applies to all airs of consecutive rows, each row has to be numerically naller than the preceding one, and hence no two rows n be alike. Thus, it is always possible to reduce any attern matrix to one containing no more than $p - 1$ olumns. This reduction scheme, carried out on the set natrix (1)], is demonstrated in matrix (2); matrix (3) the resulting reduced matrix.

Nonminimal Reduction of Matrix (1)

(2)

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccccc}1&2&3&4&5&6&7&8&9\end{array} \\ \left[\begin{array}{ccccccccc}1&1&1&0&0&1&1&1&1\\1&1&1&0&0&1&0&0&1\\1&1&0&1&0&0&1&0&1\\1&1&0&1&0&0&1&0&1\\0&0&1&1&1&0&1&0&0\\0&0&0&1&0&1&1&1&0\end{array}\right] \\ \begin{array}{ccccccccc}\uparrow&\uparrow&\uparrow&&&\uparrow&&&\end{array}\end{array}$$

(3)

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} 1237 \\ \left[\begin{array}{c}1111\\1110\\1101\\1011\\0011\\0001\end{array}\right]\end{array}$$

To verify that the upper limit may be reached, one has construct a $p \times (p - 1)$ irreducible pattern matrix. his can be readily done by forcing row 2 to be identical row 1 in all columns but column 1, forcing row 3 to be lentical to row 1 or 2 in all columns but column 2, and general forcing row i ($i = 2, 3, \cdots p$) to be identical to ny of the preceding rows except in column $i - 1$. The esult is a matrix in which all the rows are distinct and which two rows become identical if any one of the $p - 1$ olumns is deleted. Thus the proof of the theorem is omplete.

Reader Efficiency

For the set of p patterns in which the i'th pattern has he probability P_i, the average information rate is given by:

$$R = -\sum_{i=1}^{p} P_i \log_2 P_i \quad \text{bits/pattern.}$$

If each pattern is represented by c binary digits, the transmission rate is given by:

$$T = c \text{ binary digits/pattern.}$$

The redundancy ρ in the system can be defined by:

$$\rho = \frac{T - R}{T}$$

and the efficiency of the system by:

$$\eta = 1 - \rho = -\frac{1}{c}\sum_{i=1}^{p} P_i \log_2 P_i.$$

Using the theorem proved in the preceding section, we can write:

$$\frac{-\sum_{i=1}^{p} P_i \log_2 P_i}{p - 1} \leq \eta \leq \frac{-\sum_{i=1}^{p} P_i \log_2 P_i}{\{\log_2 p\}}.$$

This inequality yields the range of efficiencies which may be realized with a reader employing a minimal scanning path. For 10 equiprobable characters, for example, the efficiency of a minimal-scan reader will always lie between 0.37 and 0.83. Knowing the initial value for c, such information is invariably useful in estimating the advantage gained by designing such a reader. If in the above example, the character style requires a grid of 50 cells with the corresponding efficiency of 0.066, this advantage is apparent.

When c is considerably larger than p, sufficient advantage may be gained by realizing the lower-efficiency limit. When this is the case, the desired path (or paths) can be found either by the method described in the preceding section, or by a partitioning scheme described in another paper.[1] Either method is sufficiently simple to handle most practical cases without any need for machine computation. However, since these methods do not guarantee that the reduced path will contain less than $p - 1$ cells, they are seldom useful in cases where c is in the same order of magnitude as p or perhaps even smaller than p. In such cases, more complex schemes have to be employed which are capable of producing the minimal path for the given set.

A Minimal Reduction Method

Consider an arbitrary pattern matrix, and suppose that only the pattern represented by the first row is to be recognized, in which case any of the cells 1 to c may be selected as a legitimate path. Suppose now that pattern 2 is added to the set. Then each of the paths selected previously may or may not be still legitimate, depending on whether the part of pattern 2 in that path does or does not coincide with the part of pattern 1 in the same path. If such a coincidence occurs, the path in question has to be augmented with another cell which is numerically

[1] A. Glovazky, "Determination of redundancies in a set of patterns," IRE Trans. on Information Theory, vol. IT-2, pp. 151–153; December, 1956.

different in the two coincident patterns. In terms of the matrix, if at a certain path rows 1 and 2 coincide, the augmenting cell is represented by any column whose first and second digits differ. After carrying out all possible augmentations, a revised list of paths is obtained including all paths which may be used to recognize patterns 1 and 2. The next step is similar to the previous one: we add pattern 3 to the set and examine the adequacy of each path produced previously; if for a given path, row 3 coincides with any preceding row, this path is augmented with any cell represented by a column in which the two coincident rows differ. Carrying this procedure down to the last row results in a final list of paths which contains all paths adequate for the recognition of the p given patterns.

It should be noticed that if at any step of the procedure a path is found inadequate, its augmentation is always possible; this follows from the assumption that no two rows in the matrix can be identical in all their digits. Also, no more than two rows—one of which is the newly added row—can become coincident in any listed path; this follows from the fact that any path in the current list is by construction adequate for all rows previously covered.

A helpful, though not essential, operation is to eliminate from the list produced at the end of each stage all the implying paths. An implying path is a path containing a group of cells which constitutes another path in the same list. This operation not only contracts the size of the paths list at each stage, but also restricts it to paths which are irreducible; this follows from the fact that, since the paths lists are exhaustive, every path which is reducible is necessarily implying.

The testing and augmentation of scanning paths can best be done with the aid of the sum matrix, the application of which to minimal reduction of pattern matrices was first proposed by McCluskey.[2] Every row in this matrix is the sum modulo 2 of two rows in the pattern matrix. Consequently, a unity or a zero appearing in any column of a sum row, indicate that the component rows differ or agree, respectively, in that column. For our purpose we construct $p - 1$ sum matrices, of which the k'th one contains the sums of row $k + 1$ with all preceding k rows. Thus, to determine whether, in a given path, row $k + 1$ coincides with any preceding row, we test whether any row in the k'th sum matrix has zeros in all the columns representing this path. If such a row exists, then an augmenting cell is a cell represented by any column in which the row is unity.

The set of sum matrices appropriate for the pattern matrix (1) is shown below. The indexes attached to the rows indicate the component rows in matrix (1). To test, for example, the adequacy of the path 3-7 listed at the end of the second cycle of the procedure, we examine all rows in sum matrix No. 3. Row 4/1 of this matrix has zeros in both columns 3 and 7, which indicates that the path has to be augmented. Augmenting cells are either 2 or 8, since the corresponding columns have unities at row 4/1. The path 3-7, therefore, yields the augmented paths 2-3-7 and 3-7-8.

[2] E. C. Riekeman, A. Glovazky, and E. J. McClusky, "Determination of redundancies in a set of patterns," IRE Trans. on Information Theory, vol. IT-3, pp. 167–168; June, 1957.

The following is a summary of the proposed minimal reduction scheme, in a form suitable for computer programming:

Initial paths list: 1, 2, 3, $\cdots$, c

Initial value for k: 1.

1) For each path in the list produced in the previous cycle, determine whether the corresponding group of columns in the k'th sum matrix has a zero row.
2) If there is such a row, augment the path with a cell corresponding to each column in the sum matrix in which the row is unity.

Sum Matrices for Matrix (1)

$$\begin{array}{lrl} & & \ \ 123456789 \\ \text{No. 1} & 2/1 & [000000110] \end{array} \tag{4}$$

$$\text{No. 2}\quad \begin{array}{r} 3/2 \\ 3/1 \end{array} \begin{bmatrix} 001101100 \\ 001101010 \end{bmatrix}$$

$$\text{No. 3}\quad \begin{array}{r} 4/3 \\ 4/2 \\ 4/1 \end{array} \begin{bmatrix} 011101000 \\ 010000100 \\ 010000010 \end{bmatrix}$$

$$\text{No. 4}\quad \begin{array}{r} 5/4 \\ 5/3 \\ 5/2 \\ 5/1 \end{array} \begin{bmatrix} 100111001 \\ 111010001 \\ 110111101 \\ 110111011 \end{bmatrix}$$

$$\text{No. 5}\quad \begin{array}{r} 6/5 \\ 6/4 \\ 6/3 \\ 6/2 \\ 6/1 \end{array} \begin{bmatrix} 001011010 \\ 101100011 \\ 110001011 \\ 111100111 \\ 111100001 \end{bmatrix}$$

3) From the expanded path list, eliminate all implying paths.
4) If $k < p - 1$, add 1 to k and return to step 1). If $k = p - 1$, the list would contain all irreducible paths which are adequate for recognizing the p given patterns. The shortest paths in this list are the desired minimal paths.

The table shown demonstrates the procedure for reducing set (1), utilizing the sum matrices (4). The crossed-out paths are the implying paths eliminated at the end of each cycle. The resulting minimal matrices, corresponding to the shortest paths 2-4-8 and 2-6-8, are given in matrices (5) and (6).

As can be noticed, a nonimplying path in the list produced at the end of the first cycle cannot contain more than a single cell. This can be explained by the fact that it is always possible to find a single column in which rows 1 and 2 of the pattern matrix differ. Also, as shown above, no path requires more than one additional cell

Minimal Reduction of Matrix (1)

Ini-tial paths list	Cycle 1		Cycle 2		Cycle 3		Cycle 4		Cycle 5	
	Aug-menting cells	Paths list	Aug-menting cells	Paths list	Aug-menting cells	Paths list	Aug-menting cells	Paths list	Aug-menting cells	Paths list
1	78	~~17~~		37	28	237	14569	1237	none	1237
2	78	~~18~~		47	28	378	14569	~~2347~~		2357
3	78	~~27~~		67	28	247	none	2357	none	2379
4	78	~~28~~		78	2346	478	12359	~~2367~~		1378
5	78	~~37~~		38	27	267	none	2379	none	3578
6	78	~~38~~		48	27	678	12359	1378	none	3678
7	none	~~47~~		68	27	278	14569	~~3478~~		3789
8	none	~~48~~		~~78~~		~~378~~		3578	none	2347
9	78	~~57~~				~~478~~		3678	none	2457
		~~58~~				~~678~~		3789	none	2467
		~~67~~				238	14569	247	3568	~~2478~~
		~~68~~				~~378~~		1478	none	1478
		7	3468			248	none	~~2478~~		3478
		8	3467			~~478~~		3478	none	4578
		~~79~~				268	none	4578	none	4789
		~~89~~				~~678~~		3789	none	1267
								267	13489	2367
								1678	none	~~2467~~
								~~2678~~		~~2678~~
								~~3678~~		2679
								5678	12349	1678
								6789	none	~~15678~~
								1278	none	~~25678~~
								~~2478~~		~~35678~~
								2578	none	~~45678~~
								~~2678~~		~~56789~~
								2789	none	6789
								1238	none	1278
								~~2348~~		2578
								2358	none	2789
								~~2368~~		1238
								2389	none	2358
								248	none	2389
								268	none	248
										268

Minimal Forms of Matrix (1)

(5)	(6)
[248]	[268]
1 101	1 111
2 100	2 110
3 110	3 100
4 000	4 010
5 010	5 000
6 011	6 011

er cycle of the reduction procedure. These observations nd the fact that the procedure terminates after $p - 1$ ycles again verify that no minimal path can contain nore than $p - 1$ cells.

Error-Detecting Scanning Paths

The minimal-reduction scheme described in the preced-ng section guarantees that the "distance" between any wo patterns, *i.e.*, the number of differing cells, will be t least 1 in the determined paths. When the input atterns are not distorted and the scanner is ideal, this uarantee is sufficient to insure unique identification of ll the given patterns. In the "noisy" case, however, vhen either the patterns or the scanners are imperfect, t is desirable to maintain a larger distance between any wo patterns, so that one or more errors may be detected r corrected. If the distance maintained between any two patterns is at least d, then it is always possible to detect $d - 1$ errors or correct $\{d/2\} - 1$ errors in any given pattern.[3]

Our purpose in the noisy case is then to find scanning paths which are consistent with a prescribed minimum distance d. Such paths, which can always be found for $d = 1$, are not always realizable for $d > 1$. Certainly, no path consistent with a specified d can be found if the minimum distance d_c in the original pattern matrix is smaller than d; in such cases, therefore, no "noise-proofing" is possible unless an increase in c is permissible. In the following discussion we shall assume that $d \leq d_c$ and that c is fixed.

A scanning path consistent with a prescribed d can be found—if it exists—by the following scheme. Arrange the rows of the pattern matrix in a sequence of decreasing numerical value and find a reduced matrix by the method demonstrated in matrices (2) and (3). Now, ignoring rows which are already d cells away from all other rows, arrange the remaining (unreduced) part of the original matrix in a descending sequence and reduce it by the same method. Combine the two reduced matrices and again, ignoring all rows which are already d cells away from all other rows, reduce the remainder. Continue this process until all rows are at least d cells apart from each

[3] R. W. Hamming, "Error detecting and error correcting codes," *Bell Sys. Tech. J.*, vol. 29, pp. 147–160; April, 1950.

other. The columns of the matrix constructed by joining all the reduced matrices represent, then, a path consistent with the specified d. The important fact to observe here is that this procedure cannot involve more than d cycles, since each additional cycle increases all inter-pattern distances by at least 1 (except when a distance is already d). Using the theorem stating that no reduced matrix can contain more than $p - 1$ columns, we can now conclude that the minimal path consistent with a specified d can never contain more than $d(p - 1)$ cells. To verify that this limit may be reached, we can construct d matrices of the type described at the end of the second section; joining these matrices end-to-end results in a single $p \times d(p - 1)$ matrix in which the rows cannot be d cells apart unless all $d(p - 1)$ columns are maintained. In conclusion we can therefore write:

$$c_{\min} \leq d(p - 1).$$

The lower limit of $c_{\min}$ is dictated by the minimum number of cells capable of accommodating p patterns with minimal inter-pattern distance d. From recently published results,[4] this limit is given by:

$$c_{\min} \geq \{\log_2 p\} + d - 1.$$

Thus we have the general relationship

$$\{\log_2 p\} + d - 1 \leq c_{\min} \leq d(p - 1)$$

which checks with the separately-obtained result for $d = 1$.

The reduction scheme described in this section may be useful when c is considerably larger than $d(p - 1)$. In many practical cases, however, c is in the order of $d(p - 1)$ and often smaller than $d(p - 1)$, and a method of minimal reduction is desired. Such a method, revealing all minimal paths consistent with a specified d, is described in the following section.

Minimal Reduction With Prescribed d

Consider a sum matrix which contains the sums of all possible pairs of rows appearing in the original pattern matrix. In terms of this sum matrix, the problem of finding the minimal path with a prescribed d can be formulated as follows: we wish to delete the largest number of columns from the sum matrix so that no row will contain less than d unities. The procedure which effects this reduction is the following. Consider an arbitrary row in the sum matrix, assumedly containing unities in columns $c_1, c_2, \cdots, c_\alpha$. Then the minimal matrix has to contain at least column c_1, or $c_2, \cdots$, or c_α, since otherwise the row in question will appear as a zero row in the reduced sum matrix—which is prohibited for any $d > 0$. Suppose now that c_1 is the column which appears in the minimal matrix; delete this column and mark that all rows containing unities in c_1 are partially "taken care of," in the sense that now only $d - 1$ unities have to be guaranteed in each one of them. Carry out this operation with respect to each of the α columns, thus obtaining α reduced matrices at the end of the first cycle. With respect to each of the reduced matrices, carry out the same operation carried out on the original sum matrix; specifically: consider an arbitrary row, delete—one at a time—columns in which this row is unity, and mark the number of unities which have still to be "taken care of" in each row. The result is again an enlarged set of matrices which can be processed in the same manner. If, at the end of a cycle, it is observed that any row has been completely "taken care of"—*i.e.*, that in the columns already deleted from the corresponding matrix this row already has d unities—the row may be deleted. The entire process ends when a matrix is first obtained in which all remaining rows may be deleted; the columns which are absent from this matrix represent the cells of the desired minimal path. Since no row is omitted from a matrix before its portion in the deleted columns contains d unities, the resulting path indeed features a minimal interpattern distance of d. It can be seen that the various groups of deleted columns which are obtained at the end of each reduction cycle must include the group contained in the minimal matrix, since otherwise a zero row would appear in the minimal sum matrix. Consequently, the cycle producing the first minimal path is also the cycle producing all minimal paths.

A step which is not essential, but in most cases helpful, is to select the row which contains the smallest number of unities as the "arbitrary" row referred to above. This step will reduce to the minimum the number of matrices which have to be considered at each cycle, and will often accelerate the reduction process. Another helpful step is to eliminate all matrices whose rows and columns appear in other matrices. Other procedural details can be best explained by means of an example.

Matrix (7) contains all the rows of the matrices (4), arbitrarily numbered, and hence is the complete sum matrix of the pattern set (1). The "d_0" number attached to each row indicates how many unities in the corresponding row still have to be "taken care of." At the outset, of course, all these numbers are d which, in our example, was chosen to be 2. Row 1 (marked with an arrow), which contains the least number of unities, is seen to have unities in columns 7 and 8 of matrix (7). When column 8 is deleted, matrix (8) results, while when column 7 is deleted, matrix (9) results. The auxiliary column, now called "d_1," is modified to allow for the unities which already have been "taken care of" by the column deletion. Thus, a d_1 number attached to a row which contained a unity at the deleted column, equals the corresponding d_0 number diminished by 1; all the other d_1 numbers are the same as the corresponding d_0 numbers. Consequently, the entire d_1 column can be obtained simply by subtracting, term by term, the deleted column from the d_0 column. Proceeding with the reduction, the rows to be considered in the new matrices are again the ones containing the least number of unities—namely row 1 in matrix (8) and

[4] D. D. Joshi, "A note on the upper bounds for minimal-distance codes," *Information and Control*, vol. 1, no. 3, pp. 289–295; September, 1958.

Reduction of Matrix (1) with d = 2

The sum matrix

(7)

	123456789	d_0
→1	000000110	2
2	001101010	2
3	010000010	2
4	110111011	2
5	111100001	2
6	001101100	2
7	010000100	2
8	110111101	2
9	111100111	2
10	011101000	2
11	111010001	2
12	110001011	2
13	100111001	2
14	101100011	2
15	001011010	2

Cycle 1

(8)

	12345679	d_1
→1	00000010	1
2	00110100	1
3	01000000	1
4	11011101	1
5	11110001	2
6	00110110	2
7	01000010	2
8	11011111	2
9	11110011	1
10	01110100	2
11	11101001	2
12	11000101	1
13	10011101	2
14	10110001	1
15	00101100	1

(9)

	12345689	d_1
→1	00000010	1
2	00110110	2
3	01000010	2
4	11011111	2
5	11110001	2
6	00110100	1
7	01000000	1
8	11011101	1
9	11110011	1
10	01110100	2
11	11101001	2
12	11000111	2
13	10011101	2
14	10110011	2
15	00101110	2

Cycle 2

(10)

	1234569	d_2
1	0000000	0
2	0011010	1
→3	0100000	1
4	1101111	1
5	1111001	2
6	0011010	1
7	0100000	1
8	1101111	1
9	1111001	0
10	0111010	2
11	1110101	2
12	1100011	1
13	1001111	2
14	1011001	1
15	0010110	1

Cycle 3

(11)

	134569	d_3
→2	011010	1
3	000000	0
4	101111	0
5	111001	1
6	011010	1
7	000000	0
8	101111	0
10	011010	1
11	110101	1
12	100011	0
13	101111	2
14	111001	1
15	010110	1

Cycle 4

(12)

	14569	d_4
2	01010	0
5	11001	0
6	01010	0
10	01010	0
11	10101	0
→13	11111	2
14	11001	0
15	00110	0

(13)

	13569	d_4
2	01010	0
5	11001	0
6	01010	0
10	01010	0
11	11101	1
13	10111	1
14	11001	0
→15	01110	1

(14)

	13459	d_4
2	01100	0
→5	11101	1
6	01100	0
10	01100	0
11	11011	1
13	10111	1
14	11101	1
15	01010	0

Cycle 5

(15)

	4569	d_5
→13	[1111]	1

(16)

	1569	d_5
→13	[1111]	1

(17)

	1469	d_5
→13	[1111]	1

(18)

	1459	d_5
→13	[1111]	1

(19)

	1456	d_5
→13	[1111]	1

(20)

	1569	d_5
11	1101	0
→13	1111	1
15	0110	0

(21)

	1369	d_5
11	1101	0
13	1011	0
15	0110	0

(22)

	1359	d_5
→11	1111	1
13	1011	0
15	0110	0

(23)

	3459	d_5
5	1101	0
11	1011	0
13	0111	0
14	1101	0

(24)

	1459	d_5
5	1101	0
11	1011	0
→13	1111	1
14	1101	0

(25)

	1359	d_5
5	1101	0
→11	1111	1
13	1011	0
14	1101	0

(26)

	1345	d_5
5	1110	0
11	1101	0
13	1011	0
14	1110	0

Minimal Forms of Matrix (1) with d = 2

(27)

	24578
1	10011
2	10000
3	11010
4	00010
5	01110
6	01011

(28)

	12678
1	11111
2	11100
3	11010
4	10110
5	00010
6	00111

(29)

	26789
1	11111
2	11001
3	10101
4	01101
5	00100
6	01110

row 1 in matrix (9). These, in turn, call for the deletion of columns 7 and 8 from matrix (8) and matrix (9), respectively. The resulting reduced matrices happen to be identical, and are represented by matrix (10). Producing the auxiliary column d_2 from d_1 in the same manner that d_1 was produced from d_0, it is seen that the d_2 numbers for rows 1 and 9 of matrix (10) are zero. It can be concluded that these rows are completely "taken care of" and hence may be deleted from the matrix. Subsequent cycles are carried out in the same manner until, as in the fifth cycle of our example, matrices are obtained with an auxiliary column which is composed of zeros only. Such matrices in the example are (21), (23) and (26). The groups of columns which are absent from these matrices represent the sought scanning paths, guaranteeing at least one error-detection capability. Matrices (27)–(29) are the corresponding minimal pattern matrices.

The following summarizes the procedure, again in a form suitable for computer programming:

Initial set of matrices: the complete sum matrix.

The d_0 column: d for all rows.

Initial value for k: 1.

1) For each matrix in the set produced in the preceding cycle: a) select the row which contains the least number of unities; b) produce new matrices by deleting, one at a time, columns in which the selected row is unity; c) for each matrix thus produced, determine the d_k column by subtracting from the d_{k-1} column of the originating matrix the column which was deleted.
2) From the new set of matrices, eliminate all duplicate matrices.
3) Test whether any d_k column consists entirely of zeros. If such a column exists, the columns which are absent from the corresponding matrix represent the desired minimal path.

4) If the test 3) is negative, delete all rows whose d_k numbers are zero; add 1 to k and return to step 1).

This procedure may, of course, be used in the special case $d = 1$, for which another procedure has already been described. The advantage of the other procedure is that all irreducible paths—minimal and nonminimal—are produced simultaneously, which may be desirable from the designer's point of view. The disadvantage is that the method cannot be readily extended to cases where $d > 1$.

Conclusion

The theorems and algorithms developed in the preceding sections are valuable in the design of a minimum-scan pattern-recognizer which is to operate with a fixed and highly-standardized set of input patterns. A familiar example is a reading system for standard alpha-numerical characters, the potentialities of which in the field of commercial communication are apparent.

It should be pointed out that although the material presented above is concerned with visual patterns only, the results obtained may be directly applied to any finite set of binary messages. The problems encountered in the design of an identifying device for such messages is exactly the same as those encountered in designing a "reader" for pattern-recognition systems.

Acknowledgment

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On the Mean-Square Noise Power of an Optimun Linear Digital Filter for Correlated Noise Input*

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Summary—An asymptotic solution for the mean-square output noise power of an optimum digital filter is obtained. It is assumed that the input consists of a polynomial plus correlated noise. The asymptotic solution is found by fixing the interval between samples and allowing the number of samples to approach infinity. The solution obtained for the minimum variance filter is compared with the solution as obtained for the "least-squares filter," and it is shown that the latter filter is asymptotically efficient as compared to the former. It is shown for each of the above filters that the mean-square output noise power is proportional to the spectral density function of the correlated noise, evaluated at zero frequency, and that the factor of proportionality is the same.

THIS paper presents an asymptotic solution for evaluating the output mean-square noise power of an optimum linear digital filter when the input consists of a polynomial of degree n plus correlated noise. The asymptotic solution is obtained by fixing the interval between samples (T) and letting the finite memory of the filter (mT) approach an infinite memory filter as the number of input samples ($m + 1$) is allowed to approach infinity.

Two classes of input functions are considered as follows: a) all polynomials of degree n plus uncorrelated noise of mean square given by σ_N; b) all polynomials of degree n plus correlated noise of mean square given by σ_N.

Two optimum digital filters are discussed as follows: A) a least-squares linear digital filter which is optimum with respect to the class of input functions a; B) a minimum variance linear digital filter which is optimum with respect to the class of input functions b.

Three combinations of input functions (lower case) and filters (upper case), and the corresponding output mean square noise powers at time mT were investigated. They are:

1) Case aA with output mean square given by σ_m^2 (see Blum[1])
2) Case bB with output mean square given by $\bar{\sigma}_m^2$
3) Case bA with output mean square given by $\hat{\sigma}_m^2$.

The following theorems are proven:

I. If $N(t)$ is a random process whose spectral density function $f(\lambda)$ is piecewise continuous and positive, then

$$\lim_{m\to\infty} \frac{\hat{\sigma}_m^2}{\bar{\sigma}_m^2} = 1.$$

This theorem shows that the least-squares filter is asymptotically efficient as compared to the minimum variance filter.

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[1] M. Blum, "On the mean-square noise power of an optimum linear discrete filter operating on polynomial plus white noise," IRE Trans. on Information Theory, vol. IT-3, pp. 225–231; December, 1957.

II. The asymptotic value for $\tilde{\sigma}_m^2$ is given by

$$\lim_{m\to\infty} \frac{\tilde{\sigma}_m^2}{\sigma_m^2} = \frac{2\pi}{T} f(0).$$

Let the input to a digital filter be sampled from the function

$$e(t) = P(t) + N(t) \tag{1}$$

where

$$P(t) = \sum_{k=0}^{n} a_k P_k(t) \qquad t \geq 0$$
$$= 0 \qquad t < 0 \tag{2}$$

is a nonrandom regression function whose components[2] $P_k(t)$, $k = 0, 1, 2, \cdots n$ are known, but whose parameter vector $a = (a_0, \cdots a_n)$ is not known, and $N(t)$ is a stationary random noise process whose ensemble average is zero for all t and whose correlation function is given by[3]

$$E\{N(t)N(s)\} = \sigma_N^2 \rho(t - s), \qquad \rho(0) = 1. \tag{3}$$

Define a linear digital filter[4] with weighting sequence W_u. Then the output at $t = m$ is given by,[5] $(m = 0, 1, 2, \cdots)$

$$e_m^* = \sum_{u=0}^{m} W_u e_{m-u} \tag{4}$$

where $W_u = 0$, for $u < 0$, and $u > m$.

The output at time $t = m$ is an estimate of a desired output

$$S_m^* = \int_{-\infty}^{\infty} k(\tau) P(m - \tau)\, d\tau, \tag{5}$$

where $k(\tau)$ is defined by the desired linear operation on the input information $P(t)$. Then the optimum weighting sequence $\tilde{W}$ in the minimum variance sense, *i.e.*,

$$E\{e_m^* - S_m^*\} = 0$$
$$E\{e_m^* - S_m^*\}^2 = \tilde{\sigma}_m^2 = \text{a minimum,} \tag{6}$$

is given by the matrix equation[6,7]

$$\tilde{W} = V^{-1}P'[PV^{-1}P']^{-1}Q, \tag{7}$$

where V is the $(m + 1) \times (m + 1)$ symmetric correlation matrix with elements in the ith row and jth column given by

$$V_{ij} = \rho(i - j), \qquad i, j = 1, 2, \cdots, m + 1. \tag{8}$$

P is the $(n + 1) \times (m + 1)$ matrix whose element in the ith row and jth column is

$$P_{i-1}[m - (j - 1)] \qquad \begin{array}{l} i = 1, 2, \cdots, n + 1 \\ j = 1, 2, \cdots, m + 1, \end{array} \tag{9}$$

and Q is the column vector given by Blum's (11)[7]

$$Q_j = \int_{-\infty}^{\infty} k(\tau) P_j(m - \tau)\, d\tau, \quad j = 0, 1, 2, \cdots, n. \tag{10}$$

A second filter is considered which will be noted as the least-squares filter which is optimum for uncorrelated noise such that the matrix V becomes an identity matrix I, and (7) becomes

$$\hat{W} = P'[PP']^{-1}Q \tag{11}$$

with the corresponding output mean square $\hat{\sigma}_m^2$ for correlated noise input (case bA). Let

$$\tilde{\delta}_m^2 = \tilde{\sigma}_m^2/\sigma_N^2,$$
$$\hat{\delta}_m^2 = \hat{\sigma}_m^2/\sigma_N^2. \tag{12}$$

Then the following theorems will be proved:

Theorem I

If $P_k(t)$ is a polynomial of degree k defined over the interval $(0, m + \alpha)$, and $N(t)$ is a random process whose spectral density is piecewise continuous and positive, and α is the prediction interval of the output of the digital filter, then

$$\lim_{m\to\infty} \frac{\hat{\delta}_m^2}{\tilde{\delta}_m^2} = 1. \tag{13}$$

Theorem II

If δ_m^2 is the ratio of the mean square output noise to mean square input noise for the least squares filter, when the input noise is uncorrelated (*e.g.*, $V = I$, case aA), then

$$\lim_{m\to\infty} \frac{\tilde{\delta}_m^2}{\delta_m^2} = 2\pi f(0)$$

where $f(0)$ is the spectral density of the process $N(t)$ at zero frequency, *e.g.*, such that

$$\rho_v = \int_{-\pi}^{+\pi} e^{iv\lambda} f(\lambda)\, d\lambda, \qquad \rho_0 = 1.$$

Proof of Theorems

Let us evaluate the various mean square error ratios

$$\tilde{\delta}_m^2 = \tilde{W}'V\tilde{W}$$
$$= Q'[PV^{-1}P']^{-1}Q \tag{15}$$

$$\hat{\delta}_m^2 = \hat{W}'V\hat{W}$$
$$= Q'[PP']^{-1}PVP'[PP']^{-1}Q \tag{16}$$

$$\delta_m^2 = \hat{W}'\hat{W}$$
$$= Q'[PP']^{-1}PP'[PP']^{-1}Q = Q'[PP']^{-1}Q. \tag{17}$$

[2] The equations for the weighting sequences (7), (11) and mean square output ratios (15), (16), and (17) are valid for a general class of regression function. In order to prove the theorems, the $P_k(t)$ will be taken as orthogonal polynomials with respect to summation at the sampled points.

[3] The operator $E\{\ \}$ denotes the ensemble average of the bracketed quantities.

[4] H. M. James, N. B. Nichols, and R. S. Phillips, M.I.T. Rad. Lab. Series, McGraw-Hill Book Co., Inc., N. Y., vol. 25; 1947.

[5] Note that the sampling interval T will be taken as unity to simplify the notation. The effects of $T \neq 1$ will be considered later.

[6] The notation: -1 indicates an inverse matrix, and a prime the transpose matrix.

[7] M. Blum, "An extension of the minimum mean square prediction theory for sampled input signals," IRE Trans. on Information Theory, vol. IT-2, pp. 176–184; September, 1956.

The theorems as stated apply to regression functions $P_k(t)$ which are polynomials of degree k. The solution will be considerably simplified if the sampled polynomials are taken as orthogonal with respect to summation.[1,8,9] Let

$$P_k(u) = \frac{\xi_k(m-u)}{\sqrt{S_k}} \quad \left.\begin{matrix} u = 0, 1, 2, \cdots, m \\ k = 0, 1, 2, \cdots, n \end{matrix}\right. \tag{18}$$

where

$$\sum_{u=0}^{m} \xi_k(m-u)\xi_j(m-u) = \delta_{kj} S_k \tag{19}$$

and

$$\begin{aligned} \delta_{kj} &= 1, \qquad k = j \\ &= 0, \qquad k \neq j. \end{aligned} \tag{20}$$

Then the matrix

$$PP' = I \tag{21}$$

and (15)-(17) may be written

$$\left.\begin{aligned} \tilde{\delta}_m^2 &= Q'[PV^{-1}P']^{-1}Q \\ \hat{\delta}_m^2 &= Q'PVP'Q \\ \delta_m^2 &= Q'Q = \sum_{k=0}^{n} Q_k^2 \end{aligned}\right\}. \tag{22}$$

The proof of the theorems depends on evaluating

$$\lim_{m\to\infty} \{[PV^{-1}P']^{-1} \text{ and } [PVP']\}. \tag{23}$$

These limits have been evaluated by Grenander and Rosenblatt[10] as follows: Since

$$\lim_{m\to\infty} R_h^{(r,s)} = \sum_{u=0}^{m} \frac{\xi_r(m+h-u)\xi_s(m-u)}{\sqrt{S_r S_s}}$$

$$\begin{aligned} r, s &= 0, 1, \cdots, n \\ h &= 0, \pm 1, \pm 2, \cdots \end{aligned} \tag{24}$$

$$\xi_r(u) = 0, \qquad u < 0$$

is defined for all integral h, then define a sequence of $(n+1) \times (n+1)$ matrices R_h with elements $R_h^{(r,s)}$. A spectral distribution function $M(\lambda)$ of the regression vectors may be defined by

$$R_h = \int_{-\pi}^{+\pi} e^{ih\lambda}\, dM(\lambda) \qquad h = 0, \pm 1, \pm 2, \cdots \tag{25}$$

[8] R. L. Anderson, and E. E. Houseman, "Tables of orthogonal polynomials values extended to $N = 104$," *Res. Bull.*, 297 Iowa State College of Agriculture and Mechanical Arts, Ames; pp. 297; April, 1942.

[9] F. E. Allen, "The general form of the orthogonal polynomials for simple series with proofs of their simple properties," *Proc. Roy. Soc. Edinburgh*, vol. 50, pp. 310–320; 1935.

[10] U. Grenander, and M. Rosenblatt, "Statistical Analysis of Stationary Time Series," John Wiley and Sons, Inc., N. Y., ch. 7; 1957.

When $h = 0$, then

$$R_0 = M(\pi) - M(-\pi) \equiv M$$

and is nonsingular. Then it is shown that[10]

$$\lim_{m\to\infty} [PV^{-1}P']^{-1} = 2\pi\left\{\int_{-\pi}^{+\pi} \frac{1}{f(\lambda)}\, dM(\lambda)\right\}^{-1}, \tag{26}$$

and

$$\lim_{m\to\infty} [PVP'] = M^{-1}2\pi \int_{-\pi}^{+\pi} f(-\lambda)\, dM(\lambda) M^{-1}. \tag{27}$$

It will be shown that

$$\begin{aligned} &\text{a) } R_h^{(r,s)} = \delta_{r,s} \\ &\text{b) } dM(\lambda) = \delta(\lambda)I, \ (\delta(\lambda) \text{ is the Dirac function}) \\ &\text{c) } M = I. \end{aligned} \tag{28}$$

By (19),

$$R_0^{(r,s)} = \delta_{r,s} \quad \therefore \quad M = I. \tag{29}$$

Let $\xi_r(m+h-u)$ be expanded in a Taylor series where the polynomials $\xi_r(m-u)$ are defined continuously over the interval $[0, (m+h)]$

$$\xi_r[(m-u)+h] = \xi_r(m-u) + h\xi_{r(1)}(m-u) + \cdots + \frac{h^r}{r!}\xi_r^{(r)}(m-u). \tag{30}$$

Each term $\xi_r^{(j)}(m-u)$ is a polynomial in (m) of degree $(r-j)$, while

$$S_r = L_r \prod_{j=-r}^{r} (m-j)$$

is a polynomial in m of degree $2r+1$. Thus the limit $m \to \infty$ of each term of the form

$$\frac{h^j}{j!} \sum_{u=0}^{m} \frac{\xi_s(m-u)\xi_r^{(j)}(m-u)}{\sqrt{S_s S_r}} = 0(m^{-j}), \qquad j = 0, 1, 2, \cdots, r \tag{31}$$

for finite h. Noting that $\xi_r(u) = 0$ for $u < 0$, then $R_h^{(r,s)}$ is evaluated as follows:

$$\lim_{m\to\infty} R_h^{(r,s)} = \sum_{u=0}^{m} - \sum_{u=m-(h+1)}^{m} \cdot\left[\frac{\xi_s(m-u)}{\sqrt{S_s}} \sum_{j=0}^{r} \frac{\xi_r^{(j)}(m-u)h^j}{\sqrt{S_r}\, j!}\right]. \tag{32}$$

The first term in (32) is expanded into

$$\sum_{u=0}^{m} = \delta_{sr} + \sum_{j=1}^{r} \frac{h^j}{j!} \sum_{u=0}^{m} \frac{\xi_s(m-u)\xi_r^{(j)}(m-u)}{\sqrt{S_s S_r}}. \tag{33}$$

The components of the second term of (33) are seen to be zero in the limit, since

$$\sum_{u=0}^{m} \frac{\xi_s(m-u)\xi_r^{(j)}(m-u)}{\sqrt{S_s S_r}} = 0 \quad \text{if} \quad r \leq s \tag{34}$$

ecause of the orthogonality properties, since $\xi_r^{(j)}$ is a olynomial in $(m - u)$ of order $r - j$ and can have no omponents of the power $(m - u)^s$, and the summation s zero when $r > s$ in the limit as $m \to \infty$ by (31), since ≥ 1.

The second component of (32) can be shown to approach ero in the limit as $m^{-(1+j)}$ because of the finite number of erms $(h + 1)$ in the summation. Thus one obtains $\mathfrak{R}_h^{(r,s)} = \delta_{r,s}$ for each finite h, and the corresponding egression spectrum consists of only the one point $\lambda = 0$, o that

$$dM(\lambda) = \delta(\lambda)I. \tag{37}$$

Thus (26) and (27) are reduced to

$$\lim_{m\to\infty} [PV^{-1}P']^{-1} = 2\pi f(0)I$$

nd (38)

$$\lim_{m\to\infty} [PVP'] = 2\pi f(0)I$$

o that (22) is given by

$$\lim_{m\to\infty}\left[\frac{\tilde{\delta}_m^2}{\delta_m^2} = \frac{\hat{\delta}_m^2}{\delta_m^2}\right] = 2\pi f(0), \qquad \delta_m^2 = \sum_{k=0}^{n} Q_k^2, \tag{39}$$

nd both theorems are proved. As an example, let $N(t)$ e an uncorrelated stationary random process, with ariance σ_N^2. Then in terms of the normalized ratio δ_m^2, ne may equate the normalized correlation

$$\rho_0 = \int_{-\pi}^{+\pi} f(\lambda)\, d\lambda = 1,$$

nd since $f(\lambda)$ is a constant for all λ, $(-\pi \leq \lambda \leq +\pi)$, then

$$f(0) = \frac{1}{2\pi}$$

and

$$\lim_{m\to\infty}\left[\frac{\tilde{\delta}_m^2}{\delta_m^2} = \frac{\hat{\delta}_m^2}{\delta_m^2}\right] = 1.$$

When the linear operator $k(\tau)$ is defined so that the desired output is an estimate of the Mth derivative evaluated at $m + \alpha$, then it is shown in a previous paper[1] that the exact expression for δ_m^2 is given by

$$\delta_m^2 = \sum_{L=M}^{n} \frac{[\xi_L^{(M)}(m + \alpha)]^2}{S_L}. \tag{40}$$

Tables and Graphs of δ_m^2 are given in the previous paper[1] for zero, first and second derivatives, for $\alpha = 0$ (zero lag), and for values of m up to 1000.

A comparison with an asymptotic expression for δ_m^2 derived by Johnson[11] is given to enable one to determine how large m should be before placing confidence in the asymptotic expression. Additional work being done on evaluating the mean-square error output for continuous filters indicates that the finite memory of the filter mT (where T is taken as the interval between samples) must be large compared to the effective time constant B of the correlation function for the asymptotic properties to be valid. The effects of nonunity sampling interval $(T \neq 1)$ are to replace π by π/T in (26) and (27) and the final solution (39), *e.g.*,

$$\lim_{m\to\infty}\left[\frac{\tilde{\delta}_m^2}{\delta_m^2(T)} = \frac{\hat{\delta}_m^2}{\delta_m^2(T)}\right] = \frac{2\pi f(0)}{T}. \tag{41}$$

For example, if

$$\rho(\tau) = e^{-|\tau|/B},$$

then $mT/B \gg 1$ for (41) to hold. The effects of $T \neq 1$ on δ_m^2 for the derivative operator are discussed in a previous paper.[1]

[11] K. R. Johnson, "Optimum, linear, discrete filterings of signals containing a nonrandom component," IRE Trans. on Information Theory, vol. IT-2 pp. 49–55; June, 1956.

Single Error-Correcting Codes for Asymmetric Binary Channels*

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Summary—In a highly-asymmetric binary channel it may be necessary to correct only those errors which result from incorrect transmission of one of the two code elements. Minimum weight-distance relationships and rules for generating single-error correcting codes in such situations are given. More code characters are generally obtained for a given character length than are obtained with codes designed for single-error correction in symmetric channels. Examples are given, including one which specifies the code which results in the highest average probability of correct transmission of equiprobable messages through a highly-asymmetric channel.

Introduction

Let us consider the memoryless binary channel shown in Fig. 1 where

$\alpha \triangleq \Pr\{\text{any received 0 is delivered as a 0}\}$, and

$\beta \triangleq \Pr\{\text{any received 1 is delivered as a 0}\}$.

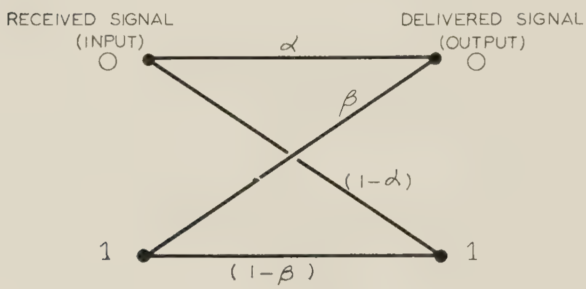

Fig. 1—Memoryless binary channel; $0 \leq \alpha, \beta \leq 1.0$.

Silverman[1] has shown that a channel characterized by α and β may be used to transmit information if $\alpha \neq \beta$, but that channel capacity is very low unless $\alpha \gg \beta$ or $\beta \gg \alpha$. (For example, if $\alpha = 0.6$ and $\beta = 0.3$, the channel capacity is 7 per cent of that of a noise-free channel transmitting the same average number of symbols per second.) For the present we will assume only that the relationship between output (delivered) signals and code elements is such as to make $\alpha > \beta$.

If in transmitting a code character, k of the received ones are delivered as zeros and l of the received zeros are delivered as ones, we will say that both a *k-tuple 1-error and an l-tuple 0-error* have occurred. When the channel has symmetric transmission properties, the general requirement that $\alpha \neq \beta$ may be reduced to $\alpha > 0.5$. As no distinction is made between 0-errors and 1-errors in a symmetric channel, it follows that all k-tuple errors are equally probable and $(k + 1)$-tuple errors are less likely than k-tuple errors.

In asymmetric channels, however, $(k + 1)$-tuple 1-errors may be more probable than k-tuple 0-errors. For example, 110 will more likely be received as 000 than as 111, provided

$$\alpha\beta^2 > (1 - \alpha)(1 - \beta)^2, \text{ or } (1 - \alpha) < \frac{\beta^2}{1 - 2\beta(1 - \beta)}. \quad (1)$$

In what follows, it will be assumed that the channel is highly-asymmetric with $\beta \gg (1 - \alpha)$.[2]

Error-Detecting and -Correcting Codes

In the coding problem we shall discuss, character length and error tolerances are specified and we seek to maximize the number of different messages which may be sent subject to these conditions. This is essentially the problem considered by Hamming[3] and Golay[4] among others, for a symmetric channel. In the symmetric channel case, error tolerances result in requirements such as "correct all single errors and detect all double errors." For highly-asymmetric channels, however, the resultant requirements will be of forms such as "correct all single 1-errors and detect all double 1-errors."

Hamming established the minimum distance requirements between input code characters for error-detecting and-correcting codes. These are sufficient to insure k-tuple error correctability or detectability in any type of channel but may be relaxed somewhat when only k-tuple 1-errors are to be considered. The nature of this relaxation is easily seen in the case where α and β are such that we may neglect 0-errors and correct only single 1-errors. Pairs of code characters must be at least three-distant when all single errors are to be corrected but, in the case of single 1-error

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[1] R. A. Silverman, "On binary channels and their cascade," IRE Trans. on Information Theory, vol. IT-1, pp. 19–27; December, 1955.

[2] Operating experience has shown that certain magnetic tape-storage units exhibit just such transmission properties.

[3] R. W. Hamming, "Error-detecting and error-correcting codes," *Bell System Tech. J.*, vol. 29, pp. 147–160; April, 1950.

[4] M. J. E. Golay, "Binary coding," IRE Trans. on Information Theory, vol. IT-4, pp. 23–28; September, 1954.

orrection, pairs of code characters need only be two-istant provided they are of the form $00XXX$ and $11XXX$. Thus when $10XXX$ or $01XXX$ is delivered in the asymmetric case, it is assumed that the input code character was $11XXX$.

If $W(x)$ is the weight (number of 1's) of code character and $D(x, y)$ is the Hamming distance between code haracters x and y, then we may use the fact that

$$D(x, y) \geq | W(x) - W(y) | \qquad (2)$$

o establish the requirements for single 1-error correctability in the form

$$D(x, y) \geq 3, \quad \text{or} \qquad (3)$$

$$| W(x) - W(y) | \geq 2, \qquad (4)$$

or all x and y. Any weaker requirement such as (4) may llow an increase in the number of different sequences which can be sent with a particular character length. We will describe a coding scheme below which makes use of (4) to achieve such an increase.

When β is much greater than $(1 - \alpha)$, error tolerances may require us to correct k-tuple 1-errors while neglecting all 0-errors. The minimum weight-distance requirements for pairs of code characters of a k-tuple 1-error correcting ode are

$$| W(x) - W(y) | \geq k + 1, \quad \text{or} \qquad (5)$$

$$D(x, y) \geq 2(k + 1) - | W(x) - W(y) |, \qquad (6)$$

or all x and y where $|W(x) - W(y)| = 0, 1, \cdots, k$.[5]

Single 1-Error Correction

Let us assume for the moment that the characters of he single 1-error correcting code are to be of even length, $2m$. We first specify code character prefixes of length m by forming all possible m-length binary sequences. For example, if $2m = 6$, the prefixes would range from 000 to 111. Suffixes are generated for a given prefix by adding hat prefix to code characters of an m length single error-correcting code. The addition is performed position by position, modulo two, and the m-length code is taken to be a single-error-correcting Hamming code formed with even parity checks. Thus, when $m = 3$, 000 and 111 are he single-error-correcting code characters and 110, as a typical prefix, will be combined with suffixes 110 and 001 to give code characters 110110 and 110001.

In order to detect and correct errors at the output of he channel, we add the prefix of the delivered signal to he suffix and test to see if the sum is one of the code characters used to generate suffixes. For example, if 110100 is delivered by the channel, we take

$$\begin{array}{r} 110 \\ \oplus\, 100 \\ \hline 010 \end{array} \qquad (7)$$

and deduce that an error has occurred in the second or fifth position. As the channel is such that 0-errors may be neglected, we assume that a 1-error has occurred and that the input signal was 110110. Note that the symbol-correcting nature of the correction scheme depends upon the use of an m-length symbol correcting code in generating suffixes. It is for this reason that the Hamming Code was chosen even though it is sometimes possible to generate more suffixes for a given prefix through the use of single-error message-correcting codes.[4]

The rules of generation are explicitly stated below and are followed by two examples of code generation. The proof that all pairs of code characters so generated satisfy (3) or (4) is included as the Appendix.

Let us use the following notation:

- n : code character length: $n > 1$.
- m : prefix length:
  - $m = n/2$ when n is even.
  - $m = (n - 1)/2$ when n is odd.
  - Suffix is, therefore, of length $(n - m)$.
- $[H]$: set of all single-error-correcting Hamming code characters of length $(n - m)$ formed with even parity checks. H_0 is that element of $[H]$ consisting of $(n - m)$ 0's. When $(n - m) = 1$ or 2, $[H] \triangleq H_0$.
- h_{n-m}: number of elements in $[H]$.
- $\oplus$: position-by-position addition, modulo two.

The rules of generation are then

1) Form all binary characters of length m. These are the prefixes.

2) Supply suffixes for each prefix of even weight, Pe_i, by forming $(Pe_i \oplus H_i)$ for each element of $[H]$. When n is odd, append a 0 to the right end of Pe_i before adding.

3) Supply one suffix for each prefix of odd weight, P_{0i}, by forming $(P_{0i} \oplus H_0)$.

Example A: $n = 8$, $m = 4$, $h_{m-n} = 2$

$$[H]: H_0 = 0000$$

$$H_1 = 0111^6$$

[5] Further discussion of multiple 1-error correction and detection may be found in "Multi-Error-Correcting Codes for Asymmetric Channels," to be given at the International Symposium on Circuit and Information Theory, June, 1959.

[6] Hamming's double-error-detecting code may be used at this point. H_1 would then be 1111 or four-distant from H_0.

TABLE I (EXAMPLE A)

		Prefixes of Even Weight								Prefixes of Odd Weight							
		0000	0011	0101	0110	1001	1010	1100	1111	0001	0010	0100	0111	1000	1011	1101	1110
Suffixes	$\oplus$ H₀	0000	0011	0101	0110	1001	1010	1100	1111	0001	0010	0100	0111	1000	1011	1101	1110
	$\oplus$ H₁	0111	0100	0010	0001	1110	1101	1011	1000								

Thus the code words obtained with H_0 are

00000000	10011001	00010001	10001000
00110011	10101010	00100010	10111011
01010101	11001100	01000100	11011101
01100110	11111111	01110111	11101110

and those obtained with H_1 are

00000111	10011110
00110100	10101101
01010010	11001011
01100001	11111000.

Example B: $n = 9$, $m = 4$, $h_{m-n} = 4$

$$[H]: H_0 = 00000$$
$$H_1 = 00111$$
$$H_2 = 11001$$
$$H_3 = 11110$$

Example C: $n = 9$

1) 000100110 is delivered.

2)
$$\begin{array}{r} 0001 \\ \oplus 00110 \\ \hline 00100 \end{array}$$

3) Hamming check indicates error is in 3rd or $(m + 3)$rd position.

4) No sum modulo two required.

5) Correct character is 001100110.

Example D: $n = 9$

1) 000100001 is delivered.

2)
$$\begin{array}{r} 0001 \\ \oplus 00001 \\ \hline 00011 \end{array}$$

3) Hamming check indicates error is in 3rd or $(m + 3)$rd position.

4) Sum modulo two of prefix is odd.

5) Correct character is 001100001.

TABLE II (EXAMPLE B)

		Prefixes of Even Weight							
		0000	0011	0101	0110	1001	1010	1100	1111
Suffixes	$\oplus$ H₀	00000	00110	01010	01100	10010	10100	11000	11110
	$\oplus$ H₁	00111	00001	01101	01011	10101	10011	11111	11001
	$\oplus$ H₂	11001	11111	10011	10101	01011	01101	00001	00111
	$\oplus$ H₃	11110	11000	10100	10010	01100	01010	00110	00000

		Preffixes of Odd Weight							
		0001	0010	0100	0111	1000	1011	1101	1110
Suffixes	$\oplus$ H₀	00010	00100	01000	01110	10000	10110	11010	11100

Rules for Error Correction

Align the first position of the received code character with the $(m + 1)$st position. Perform a position-by-position addition modulo two and apply a single-error-correcting Hamming check on the sum. If one position is incorrect in the sum and it corresponds to the addition of a zero to a one or a one to a zero, replace the zero by a one. If the incorrect position corresponds to the addition of a zero to a zero, sum the prefix modulo two replacing the zero in the prefix if this sum is odd and replacing the zero in the suffix if this sum is even.

Number of Code Words Obtained

The above system yields $(h_{n-m} + 1)2^{m-1}$ code characters of length n where h_{n-m} is the number of code characters in a single-error-correcting Hamming code of length $(n - m)$. When n does not correspond to a close-packed[7] Hamming case, $(h_{n-m} + 1)2^{m-1}$ is equal to $(h_n + 2^{m-1})$ where h_n is

[7] In this context, close-packed implies n is of the form $2^a - 1$, $a = 0, 1, 2, \cdots$. Further discussion of close-packed codes may be found in C. Y. Lee, "Some properties of nonbinary error-correcting codes," IRE Trans. on Information Theory, vol. IT-4, pp. 77–82; June, 1958.

ie number of code words in a single-error correcting-amming code of length n. Less than h_n code characters[8] e obtained in close-packed Hamming cases, but slight odification of the above method enables h_n code charac-rs to be generated. Table III lists the number of code aracters obtained with various coding schemes for alues of n between two and sixteen.

TABLE III

Number of Code Characters in Single-Error-Correcting Codes		
Hamming Code	Single 1-Error Correcting Code	Golay Code
2	2	
2	2	
2	$2 + 2$	
2^2	$2^2 + 2$	
2^3	$2^3 + 2^2$	
2^4	$2^3 + 2^2$	
2^4	$2^4 + 2^3$	20
2^5	$2^5 + 2^3$	38
2^6	$2^6 + 2^4$	68
2^7	$2^7 + 2^4$	
2^8	$2^8 + 2^5$	
2^9	$2^9 + 2^5$	
2^{10}	$2^{10} + 2^6$	
2^{11}	$2^{10} + 2^6$	
2^{11}	$2^{11} + 2^7$	

Note that when $n = 3$, the use of $[H] \overset{\Delta}{=} H_0$ enables h_n code characters to be nerated even though this is a close-packed case. In other close-packed cases, e Hamming Code itself should be used.

Correct Transmission of Equiprobable Messages

The codes described thus far have sought to maximize ie number of possible messages for a given character ngth and error tolerance. We shall now briefly consider ie case where character length and number of messages e specified and it is desired to maximize the probability correct transmission of these messages. Slepian[9] has esigned codes for use in a symmetric channel which are ptimal in the latter sense for many combinations of iaracter length and number of messages. In example E e present one of the Slepian codes which is optimal for ur equiprobable messages and character length four. message is now said to be transmitted correctly if any the output signals assigned to it is delivered by the iannel. Note that every possible output signal has been ssigned to a message even if no one message is most likely have resulted in a particular output signal. Thus in xample E, 1010 is equally likely to have been caused by essage B or message D but has been assigned to message .

xample E: Given:

1) Symmetric channel, $\beta = (1 - \alpha)$.
2) Four equiprobable messages (A, B, C and D).
3) Code characters to be of length four.

[8] The actual number is $(\frac{1}{2}h_n + 2^{m-1})$ which can be shown to be ss than h_n.

[9] D. Slepian, "A class of binary signaling alphabets," *Bell System ch. J.*, vol. 35, pp. 203–234; January, 1956.

One of the Slepian codes which results in highest probability of correct transmission under these conditions is found as Table IV. No rules are known which will generally yield such optimal codes when $\beta \gg (1 - \alpha)$ but, in the particular case of four equiprobable messages and character length four, it can be shown by exhaustion that the code described in Example F is optimal.

TABLE IV

	Message	A	B	C	D
CODE S	Input signal assigned to the above message.	0000	1011	0101	1110
	Output signals assigned to the above message.	0000 0100 0010 1000	1011 1111 1001 0011	0101 0001 0111 1101	1110 1010 1100 0110

Example F: Given:

1) Asymmetric channel, $\beta \gg (1 - \alpha)$.
2) Four equiprobable messages (A, B, C and D).
3) Code characters to be of length four.

The code which results in highest average probability of correct transmission under these conditions is found as Table V.

TABLE V

	Message	A	B	C	D
CODE A	Input signal assigned to the above message.	0000	0101	1010	1111
	Output signals assigned to the above message	0000	0101 0001 0100	1010 0010 1000	1111 1110 1101 1011 0111
					0011 0110 1001 1100

Note that the output code characters above the dashed line correspond to those generated by the rules for a single 1-error correcting code as given above. The inclusion of the characters below the line naturally requires that the correction scheme be considerably more complex than before.

Although no general conclusions may be drawn from so specific an example, it is still of interest to investigate under what conditions Code S and Code A are of equal value. In Table VI we have assigned values of β to the symmetric channel and have then calculated values of $(1 - \alpha)$ and β for the asymmetric channel such that the probability of correctly transmitting any of the four messages through the latter using Code A is slightly higher than the probability of correctly transmitting that message through the symmetric channel using Code S.

TABLE VI

Symmetric Channel	Asymmetric Channel	
$\beta = (1 - \alpha)$	β	$(1 - \alpha)$
1.0×10^{-6}	700×10^{-6}	0.25×10^{-6}
1.0×10^{-3}	20×10^{-3}	0.25×10^{-3}
1.0×10^{-2}	7.0×10^{-2}	0.25×10^{-2}
0.1	0.25	0.032

Note that 0.25 is sufficiently greater than 0.032 to insure that the least probable double 1-error in Code A is more likely than the most probable single 0-error.

Conclusion

The techniques of establishing minimum weight-distance relationships through use of shorter error-correcting-code characters has been used to generate multiple 1-error correcting codes as indicated above. Early work indicates that the problem of designing symmetrical multiple-error-correcting codes may also be profitably approached in this way.

The choice of a code for a particular system usually involves the consideration of many factors and it is therefore generally not possible to determine under what conditions an asymmetric code is desirable. It would seem that asymmetric coding should be considered whenever prototypes exhibit highly-asymmetric transmission characteristics or whenever such characteristics depend upon the setting of a bias or threshold level.

Appendix

In order to show that the rules given above result in a single 1-error correcting code, we shall use the following relationship:

$$\left.\begin{array}{l} D(x, y) = a \\ D(x \oplus z, y \oplus w) = b \\ a, b = 0, 1, 2, \cdots, n \end{array}\right\} \text{ implies } \left\{\begin{array}{l} D(z, w) = |a - b| + 2c \\ c = 0, 1, \cdots, \text{MIN}(a, b, n - a, n - b) \end{array}\right. \tag{8}$$

where n is the length of binary characters x, y, z and w. The proof of (8) follows directly from the fact that the minimum value of $D(z, w)$ is $|a - b|$ while the maximum value is the smaller of $(a + b)$ and n. In the special case where $a = 0$, we have

$$D(x \oplus z, x \oplus w) = D(z, w). \tag{9}$$

We use (9) to show that two-code characters generated under the above rules having the same prefix satisfy (3) as all elements of $[H]$ are at least 3-distant.

If two code characters have different prefixes but both prefixes are of even weight or both are of odd weight, the prefixes must be at least 2-distant. Using (8) we can show that the suffixes of 2-distant prefixes are at least 1-distant and hence the code characters satisfy (3).

Finally we consider the case of one code character with prefix of even weight and one code character with prefix of odd weight. When these prefixes are 1-distant, (8) may be used to show that the suffixes are at least 3-distant unless they were both formed by the addition of H_0. But when H_0 has been added to both prefixes, the resultant code characters differ in weight by at least 2 and (4) is, therefore, satisfied.

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Optimal Filtering of Periodic Pulse-Modulated Time Series*

WILLIAM A. JANOS†

Summary—The present study concerns the optimal filtering of class of input time series in which the amplitude is modulated by niformly-pulsed periodic functions. A uniform sampling of the tput at a period equal to the pulsing period displays the property time invariance. The consequent usage of bilateral Fourier-place transformations and the separability of terms implicit in e time-invariant nature of the processing effectively inverts the 'iener-Hopf equation and solves the problem. The weighting nction is shown to be the sum of the Wiener-Zadeh-Ragazzini lution for the unmodulated case and set of appropriately-weighted elta-function-derivative terms occurring at each end point of the ulsing intervals.

THE title could equally well be "Optimal Pulsed-Memory Filtering of Stationary Time Series," since the form of the optimizing condition is that or a pulsed-memory filter. The difference lies in the nterpretation; the pulsed-memory filter characteristic is ero outside the pulsing intervals, whereas in the pulse-nodulated input case the weighting function is generally onzero for all positive time.

The comparative length of the present study has esulted in its presentation in three parts.

Part I—Derivation of Pulsed-Memory Wiener-Hopf ntegral Equation. This part contains the Introduction, ormulation of the Problem, and the initial step in the nalysis where the optimizing condition, in the form of the Viener-Hopf equation, is shown to involve a pulsed-nemory filter.

Part II—Derivation of Optimal Pulsed-Memory Filter. This part contains the complete analysis and inversion of he optimizing integral equation. The solution is shown to ake the form of the conventional Wiener solution for the nmodulated, unpulsed solution, plus a sum of appro-riately-weighted singularities, delta-function-derivative erms to compensate for the loss in area in the convolution ntegrand due to pulsing. A Review of Solution Procedure s included to summarize the method.

Part III—Finite Memory Considerations. This part iscusses finite memory considerations still applicable to ulsed-memory filters. An example of the infinite memory tationary input is included.

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Part I—Derivation of Pulsed-Memory Wiener-Hopf Integral Equation

Introduction

The following is an investigation of the general problem of obtaining a linear time-invariant system to perform a mean-squares "best" approximation to a desired linear operation on a periodic pulse-modulated time series. This time series will consist of a stationary, ergodic random part and, at most, a sure function made up of an arbitrary linear combination of time translation-invariant elements.

The theory used has been developed in its applicable form by Norbert Wiener [1], and extended by various others [2, 3]. Whereas previous work has been concerned with samples of the past of the input function over a continuum of time points and a discrete set of time points as separate problems, the present case has to do with a special combination of both types of sampling. It will be shown, however, that the problem is reducible to that of solving a generalized form of Wiener-Hopf integral equation. The solution technique for this equation will then depend in part upon the methods of Wiener and the others.

With the class of input functions F_i which undergo a periodic modulation, there is associated the class gF_i where g is the periodic modulating function of period T, *i.e.*,

$$g(\tau) = g(\tau + mT).$$

Consider the response of a linear time-invariant system, of impulse response $W(\tau)$, to this periodically modulated input gF_i.

$$F_0(t) = \int_{-\infty}^{t} W(t - \tau)g(\tau)F_i(\tau)\, d\tau$$

or

$$F_0(t) = \int_0^{\infty} W(\tau)g(t - \tau)F_i(t - \tau)\, d\tau.$$

Now let the input functions be shifted in time origin by mT,

$$F_i(\tau) \to F_i(\tau + mT);$$

then

$$F_0'(t) = \int_{-\infty}^{t} W(t - \tau)g(\tau)F_i(\tau + mT)\, d\tau$$

$$= \int_{-\infty}^{t+mT} W(t + mT - \tau)g(\tau - mT)F_i(\tau)\, d\tau.$$

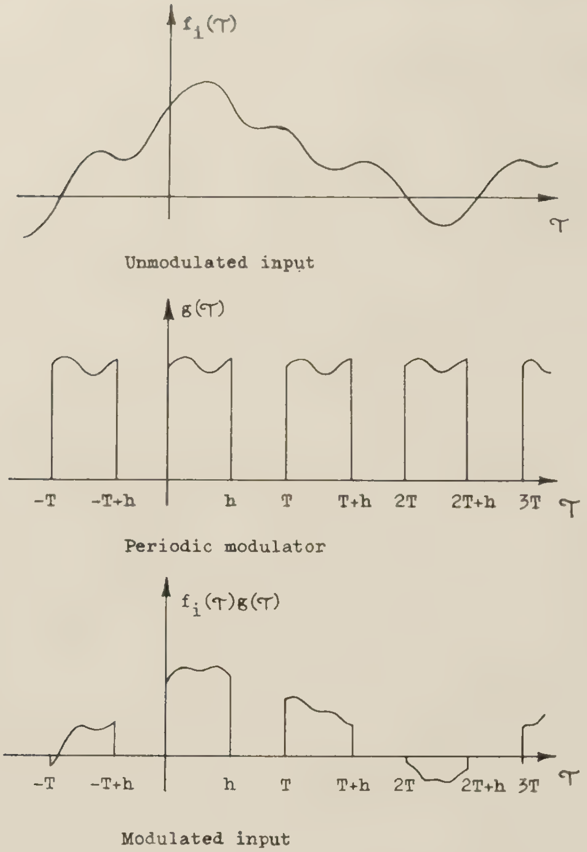

Fig. 1—Input and output waveforms of the periodic modulator.

But since $g(\tau)$ has a period T,

$$F_0'(t) = \int_{-\infty}^{t+mT} W(t + mT - \tau)g(\tau)F_i(\tau)\, d\tau = F_0(t + mT)$$

or

$$F_i(\tau + mT) \supset F_0(t + mT).$$

Therefore, the system is time-invariant with respect to input-time origin translations of mT. In order to obtain this condition in a physical system, it is necessary to sample the output uniformly with a period T.

Since the process of periodic output sampling reveals a time-invariant character, the subsequent investigation will deal only with this sampled case. It is expected that, for most cases, this possible oversimplification will be compensated for by the greater generality of the results. Hence the over-all operator is to filter a piecewise continuous time process into a discrete time-ordered sequence.

Formulation of Problem

The formulation of the optimal filter problem is the customary one [1–3]. Given an input function,

$$f_i = f_{im} + f_{in} + f_{is},$$

the true signal consists of a stationary random part, f_{im}, and a nonrandom part,

$$f_{is}(t) = \sum_1^N \alpha_k f_k(t), \quad \text{where } N \text{ and the } f_k \text{ are known.}$$

The noise, f_{in}, is assumed stationary and random. It is further assumed that all second moments of the joint processes (f_{im}, f_{in}) are known, *i.e.*,

$$\langle f_{ip}(t) f_{i0}(t - \tau)\rangle, \quad \text{for all } \tau \text{ and } P,\ 0 = m, n.$$

Introduction of the condition of zero means for f_{im}, and f_{in} amounts to a subtraction of their steady-state rms values. With this added condition, the problem is reduced to that of obtaining an optimal filter to operate on the functions f_{is} and the unpredictable parts of the input.

The optimum criterion is in the sense of minimum variance of output minus desired output.

The output is given by

$$f_0(t) = \int_{-\infty}^{t} d\tau\, W(t - \tau)g(\tau)f_i(\tau).$$

$W(\tau)$ is the impulse response (generalized one-sided retarded Green's function) of the linear time-invariant system, and $g(\tau) = g(\tau + mT)$, the periodic input modulator.

Given a desired operation on the signal part of the input,

$$f_{Ds}(t) = \int_{-\infty}^{\infty} L(\tau)\{f_{im}(t - \tau) + f_{is}(t - \tau)\}\, d\tau.$$

The error is to be minimized in the mean-square sense,

$$\begin{aligned} e(t) &= f_0(t) - f_D(t) \\ &= \int_{-\infty}^{t} d\tau\, W(t - \tau)g(\tau)f_i(\tau) \\ &\quad - \int_{-\infty}^{\infty} d\tau\, L(\tau)\{f_{is}(t - \tau) + f_{im}(t - \tau)\}. \end{aligned}$$

On taking the ensemble mean,

$$\langle e(t)\rangle = \int_{-\infty}^{t} d\tau\, W(t - \tau)g(\tau)f_{is}(\tau) - \int_{-\infty}^{\infty} d\tau\, L(t - \tau)f_s(\tau),$$

since the means of both f_m and f_n are by construction zero, *i.e.*,

$$\langle f_m(t)\rangle = \langle f_n(t)\rangle = 0.$$

Now if we set $t = mT$, T being the period of g, and then impose the condition of zero mean on e,

$$\begin{aligned} \langle e(mT)\rangle &= \int_{-\infty}^{mT} d\tau\, W(mT - \tau)g(\tau)f_{is}(\tau) \\ &= \int_{-\infty}^{\infty} d\tau\, L(mT - \tau)f_{is}(\tau) = 0, \end{aligned}$$

or

$$\begin{aligned} &\int_0^{\infty} d\tau\, W(\tau)g(-\tau)f_{is}(mT - \tau) \\ &\qquad = \int_{-\infty}^{\infty} d\tau\, L(\tau)f_{is}(mT - \tau) \qquad (1) \end{aligned}$$

$_s(\tau)$ is a linear combination of time translation-invariant ements $f_k(\tau)$, *i.e.*, $f_{is}(\tau)$ may be considered a vector in a nite dimensional function space $\mathfrak{F}_\tau$ which remains unhanged with respect to translations in time, $\mathfrak{F}_\tau = \mathfrak{F}_{\tau+mT}$. xamples of such a space are the set of all polynomials f a known degree, or more generally, the set of eigenunctions of a constant parametered differential operar [3].

Let a choice of the base vectors for a given $\mathfrak{F}_\tau$ be $_1, f_2, \cdots, f_N)$ then

$$f_{is}(mT - \tau) = \sum_1^N \alpha_k f_k(mT - \tau)$$

ut since $\mathfrak{F}_\tau$ is invariant

$$f_k(mT - \tau) = \sum_1^N \underset{kk'}{S}(-\tau) f_{k'}(mT), \quad \text{where} \quad \underset{kk'}{S}(-\tau)$$

the k, k'th matrix element of the representation of the anslation operator in $\mathfrak{F}_\tau$. Thus

$$f_s(mT - \tau) = \sum_{k=1}^N \sum_{k'=1}^N \alpha_k \underset{kk'}{S}(-\tau) f_{k'}(mT)$$

nd (1) becomes

$$\sum_{k'} \alpha_k f_{k'}(mT) \int_0^\infty d\tau\, W(\tau) g(-\tau) \underset{kk'}{S}(-\tau) = \sum_{k,k'} \alpha_k f_{k'}(mT) \int_{-\infty}^\infty d\tau\, L(\tau) \underset{kk'}{S}(-\tau)$$

hich holds in general for

$$\int_0^\infty d\tau\, W(\tau) g(-\tau) \underset{kk'}{S}(-\tau) = \int_{-\infty}^\infty d\tau\, L(\tau) \underset{kk'}{S}(-\tau), \qquad \begin{matrix} k = 1, \cdots, N \\ k' = 1, \cdots, N \end{matrix} \tag{2}$$

set of N^2 equations of constraint on $W(\tau)$.

Now we must consider the pulsed nature of $g(\tau)$. $(\tau) = g(\tau + mT)$ and $g(\tau) = 0$ for $mT \geq \tau \geq mT + h$, ll m integers, or $g(-\tau) = 0$, $mT \leq \tau \leq mT - h$. Thus is the duration of each pulse and $g(\tau)$ is represented as n infinite train of such pulses.

If $u(\tau)$ symbolizes a train of unit pulses,

$$\begin{aligned} u(\tau) &= 1,\ mT \leq \tau \leq mT + h, \quad \text{all } m \\ &= 0, \text{ otherwise.} \end{aligned}$$

hen $g(\tau) = g'(\tau)u(\tau)$ and $g'(\tau)$ can be considered equal $g(\tau)$ over the nonzero pulse intervals and continuous between. Hence (2) may have the form

$$\int_0^\infty W(\tau) g'(-\tau) u(-\tau) \underset{kk'}{S}(-\tau)\, d\tau = \int_{-\infty}^\infty L(\tau) \underset{kk'}{S}(-\tau)\, d\tau$$

nd on setting

$$\bar{W}(\tau) = W(\tau) g'(-\tau) \tag{3}$$

the equations of constraint become

$$\int_0^\infty d\tau\, \bar{W}(\tau) u(-\tau) \underset{kk'}{S}(-\tau) = \int_{-\infty}^\infty d\tau\, L(\tau) \underset{kk'}{S}(-\tau)\, d\tau$$

$$k, k' = 1, 2, \cdots, N. \tag{4}$$ [1]

On recalling the expression for the error and using the periodicity of g and definition (3),

$$e(rT) = \int_0^\infty d\tau\, \bar{W}(\tau) u(-\tau) f_i(rT - \tau) - \int_{-\infty}^\infty d\tau\, L(\tau) \{ f_{is}(rT - \tau) + f_{im}(rT - \tau) \}$$

and with (4)

$$e(rT) = \int_0^\infty d\tau\, \bar{W}(\tau) u(-\tau) \{ f_{im}(rT - \tau) + f_{in}(rT - \tau) \} - \int_{-\infty}^\infty d\tau\, L(\tau) f_{im}(rT - \tau).$$

So the error is a linear functional only of the stationary random processes f_{im} and f_{in}. Thus

$$\begin{aligned} e^2(rT) = &\int_0^\infty \int_0^\infty d\tau\, d\tau'\, \bar{W}(\tau) \bar{W}(\tau') u(-\tau) u(-\tau') \\ &\cdot \{ f_{im}(rT - \tau) + f_{in}(rT - \tau) \} \\ &\cdot \{ f_{im}(rT - \tau') + f_{in}(rT - \tau') \} \\ &- 2 \int_0^\infty \int_{-\infty}^\infty d\tau\, d\tau'\, \bar{W}(\tau) L(\tau') u(-\tau) \\ &\cdot \{ f_{im}(rT - \tau) + f_{in}(rT - \tau) \} f_{im}(rT - \tau') \\ &+ \left\{ \int_{-\infty}^\infty d\tau\, L(\tau) f_{im}(rT - \tau) \right\}^2 \end{aligned}$$

and on taking the ensemble expectation,

$$\begin{aligned} \langle e^2(rT) \rangle = &\int_0^\infty \int_0^\infty d\tau\, d\tau' \bar{W}(\tau) \bar{W}(\tau') u(-\tau) u(-\tau') \phi(\tau - \tau') \\ &- 2 \int_0^\infty \int_{-\infty}^\infty d\tau\, d\tau' \bar{W}(\tau) L(\tau') u(-\tau) X(\tau - \tau') + k_D^2 \end{aligned}$$

where $\phi(\tau)$ is the autocorrelation function of $f_{im}(\tau) + f_{in}(\tau)$, $X(\tau)$ the cross correlation between $f_{im}(\tau) + f_{in}(\tau)$ and $f_{im}(\tau')$, and k_D^2 is the mean-squared value of the desired random expression

$$\int_{-\infty}^\infty d\tau L(\tau) f_{im}(\tau' - \tau),$$

which is positive, independent of $\bar{W}(\tau)$ and constant. Due to the property of time invariance, the mean-squared error is independent of time and the output is stationary. Thus the expression to be minimized with respect to $\bar{W}(\tau)$ under the constraint (4) becomes

[1] In order for the integral on the left to converge it is sufficient that $S_{kk'}(-\tau)$ and (τ) be square integrable over $(0, \infty)$.

$$\mathcal{E}(\bar{W}) = \int_0^\infty \int_0^\infty d\tau\, d\tau' \bar{W}(\tau)\bar{W}(\tau')u(-\tau)u(-\tau')\phi(\tau - \tau')$$
$$- 2\int_0^\infty \int_{-\infty}^\infty d\tau\, d\tau' \bar{W}(\tau)L(\tau)u(-\tau)X(\tau' - \tau)$$
$$- \sum_{k,k'} \lambda_{kk'} \left\{ \int_0^\infty d\tau \bar{W}(\tau)u(-\tau) \underset{kk'}{\mathrm{S}}(-\tau) - \int_{-\infty}^\infty d\tau L(\tau) \underset{kk'}{\mathrm{S}}(-\tau) \right\}.$$

Let

$$\bar{W} \to \bar{W}_{\min} + \eta\delta\bar{W}, \quad \text{then} \quad \left\{ \frac{\partial e^2(\bar{W})}{\partial \eta} \right\}_{n=0} = 0$$

yields the minimal condition,

$$\int_0^\infty d\tau \bar{W}(\tau)u(-\tau)u(-\tau')\phi(\tau' - \tau)$$
$$= \int_{-\infty}^\infty d\tau L(\tau)u(-\tau')X(\tau' - \tau) + \sum_{k,k'} \lambda_{kk'}u(-\tau') \underset{kk'}{\mathrm{S}}(-\tau') \quad \text{for} \quad \tau' > 0 \tag{5}$$

where the $\lambda_{kk'}$ are as yet N^2 undetermined Lagrangian multipliers. The above, characteristically, is a modified Wiener-Hopf integral equation.

Analysis

Let us set

$$\psi(\tau') = \int_{-\infty}^\infty d\tau L(\tau)X(\tau' - \tau) + \sum_{k,k'} \lambda_{kk'} \underset{kk'}{\mathrm{S}}(-\tau'); \tag{6}$$

then (5) becomes

$$u(-\tau')\left\{ \int_0^\infty d\tau \bar{W}(\tau)u(-\tau)\phi(\tau' - \tau) - \psi(\tau') \right\} = 0, \quad \tau' > 0. \tag{7}$$

Notice that $\bar{W}(\tau)u(-\tau)$ is zero for all $\tau < 0$ [since $\bar{W}(\tau)$ is physically realizable] and for $mT \leq \tau \leq mT - h$, for all integers, m. Thus if we set

$$\bar{W}(\tau)u(-\tau) = \mathcal{W}(\tau), \mathcal{W}(\tau) \tag{8}$$

is a pulsed-memory filter. The solution of (5) is contained in the solution of

$$u(-\tau')\left\{ \int_0^\infty d\tau \mathcal{W}(\tau)\phi(\tau' - \tau) - \psi(\tau') \right\} = 0, \quad \text{for} \quad \tau' > 0 \tag{9}$$

where $\mathcal{W}(\tau) = 0$ for $\tau < 0$ and $mT \leq \tau \leq mT + h$, all m integers.

Part II—Derivation of Optimal Pulsed-Memory Filter

It has been shown that the optimizing condition for a uniformly-sampled output with period equal to the pulsing period T is of the form of a modified Wiener-Hopf equation involving a pulsed-memory filter, namely,

$$u(-\tau')\left\{ \int_0^\infty d\tau \mathcal{W}(\tau)\phi(\tau' - \tau) - \psi(\tau') \right\} = 0, \quad \tau' > 0 \tag{9}$$

where $\mathcal{W}(\tau) = 0$ for $\tau < 0$ and $mT \leq \tau \leq mT + h$, all m integers.

Now for most cases $\phi(\tau)$ may be considered a linear combination of exponentials, $e^{-\alpha_n|\tau|}$ so that the bilateral Laplace transform (generalized Fourier transform but for an argument phase difference of $\pi/2$) becomes

$$\int_{-\infty}^\infty d\tau \phi(\tau)e^{-p\tau} = \Phi(p) = \varphi(p)\varphi(-p) = \frac{\varphi_N(p)\varphi_N(-p)}{\varphi_D(p)\varphi_D(-p)} \tag{10}$$

where $\varphi_N(p)$ and $\varphi_D(p)$ are polynomials in p, the degree of $\varphi_D(p)$ exceeding that of $\varphi_N(p)$ at least by unity; further, both $\varphi_D(p)$ and $\varphi_N(p)$ have zeros only in the right-half complex p plane.

If $\mathcal{W}(\tau)$ were completely arbitrary for $\tau > 0$, instead of piecewise zero, the solution of (9) obtained by well-known techniques [1, 2] would be

$$''W(\tau)'' = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp e^{p\tau} \frac{1}{\varphi(p)} \left\{ \frac{\psi(p)}{\varphi(-p)} \right\}_+ = \mathcal{W}_0(\tau) \tag{11}$$

where $\Psi(p)$ is the transform of $\psi(\tau)$ and the plus subscript indicates analytic continuation in the right-half p plane of the expression within the brackets. Let us consider the transform of (9) with $\mu(-\tau)$ absent. Then,

$$Y(p)\Phi(p) - \Psi(p) = G(p) \tag{12}$$

where $G(p)$ is bounded and analytic in the left-half p plane. Or, on referring to (10) and transposing,

$$\frac{Y(p)}{\varphi_D(p)}\varphi_N(p) - \frac{\varphi_D(-p)}{\varphi_N(-p)}\Psi(p) = \frac{\varphi_D(-p)}{\varphi_N(-p)}G(p) = G'(p). \tag{13}$$

Notice that (9) and (12) imply that $G(p)$ has the poles of $1/\varphi_D(-p)$ so that $G'(p)$ can only have the poles of $1/\varphi_N(-p)$ plus, at most, those of a polynomial in $(-p)$ which can be regularized by setting

$$(-p) = \lim_{\epsilon \to 0}\left(\frac{1 - e^{\epsilon p}}{\epsilon}\right).$$

This corresponds to the limit of a finite difference in the time domain. Thus $G'_{(p)}$ is also interpreted as analytic and bounded in the left-half plane.

$Y(p)$, the transform of $\mathcal{W}(\tau)$ is zero for negative argument, $\tau < 0$, and piecewise zero for positive argument $\tau > 0$. However, $Y(p)/\varphi_{D(p)}$ is the transform of a smooth time function for positive time, $\tau > 0$, being zero for negative argument only. Thus, on taking the analytic continuation in the right-half p plane of (13),

$$\frac{Y(p)}{\varphi_D(p)} = \frac{1}{\varphi_N(p)}\left\{ \frac{\varphi_D(-p)\Psi(p)}{\varphi_N(-p)} \right\}_+. \tag{14}$$

The inverse of (14) is defined by

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp e^{p\tau} \frac{Y(p)}{\varphi_D(p)} = \mathcal{W}_1(\tau). \tag{15}$$

ow the actual solution $\mathcal{W}(\tau)$ is obtained from $\mathcal{W}_1(\tau)$ y the operator $\varphi_D(p)$, *i.e.*, $\mathcal{W}$ is the "driving function" in

$$\varphi_D \mathcal{W}_1(\tau) = \mathcal{W}(\tau). \tag{16}$$

ut over any interval of definition of (16), possibility initial conditions must be allowed. Thus, over each $\leq mT - h \leq \tau \leq mT$, $\mathcal{W}$ is nonzero and a set of initial onditions may be introduced *at* $t = mT - h$. Over $0 \leq$ $T \leq \tau \leq (m+1)T - h \mathcal{W}$ is zero, (16) is homogeneous, nplying another set of initial conditions at $\tau = mT$. hese initial conditions, manifested as δ-function derivaves at the pulse end points, must be allowable because the arbitrariness of the $\mathcal{W}_1$'s solving (15). Another ay of expressing the same thing is the following:

It is evident that $\mathcal{W}(\tau)$ depends only on that part of $\mathcal{W}_1(\tau)$ for $0 < \tau$ and $mT - h \leq \tau \leq mT$, $m > 1$, since $\mathcal{W}(\tau)$ may be expressed as $u(-\tau)\mathcal{W}_1(\quad)$, where $u(-\tau)$ a train of unit pulses which are zero over $mT \leq \tau \leq$ $T - h$. Thus, symbolically, from (13),

$$\mathcal{W}(\tau) = \varphi_D(p)\{u(-\tau)\mathcal{W}_1(\tau)\} \tag{17}$$

here the Heaviside operational notation is used to epresent a differential operator. Now the action of the perator on the expression within the brackets is of two arts, one referring to the nonzero, piecewise continuous ehavior between the discontinuities at $mT - h$ and T, and the other related to the discontinuity points hemselves. On using (15) with (17), it follows that

$$\mathcal{W}(\tau) = \frac{u(-\tau)}{2\pi i} \int_{-i\infty}^{i\infty} dp e^{p\tau} \frac{\varphi_D(p)}{\varphi_N(p)} \left\{ \frac{\varphi_D(-p)\psi(p)}{\varphi_N(p)} \right\}_+ \tag{18}$$
$$+ \sum_{m=1}^{\infty} \sum_{r=0}^{D} \{a_{mT-h,r}\delta^r(\tau - (mT - h)) + a_{mT,r}\delta^r(\tau - mT)\}$$

here the $a_{mT-h,r}$ and $a_{mT,r}$ are constants, the r denoting he rth derivative of the corresponding δ-function. The -function terms arise from differentiation of $u(-\tau)\mathcal{W}_1(\tau)$ t the discontinuities and D is the degree of $\varphi_D(p)$. The onstants are to be determined to satisfy the integral quation (9) when (16) is substituted [2].

Consider the first term on the left side of (18); by eferring to (10) and (11) we see that formally this is he optimal Wiener filter for a continuous input, then

$$\int_0^\infty \mathcal{W}_0(\tau)\phi(\tau' - \tau)\, d\tau = \psi(\tau'), \qquad 0 < \tau' \tag{19}$$

nd the integral equation (9) takes the form

$$\int_0^\infty u(-\tau)\mathcal{W}_0(\tau)\phi(\tau' - \tau)\, d\tau$$
$$+ \sum_{m=1}^{\infty} \sum_{r=0}^{D} \left\{ a_{mT-h,r}(-)^r \left(\frac{d^r}{d\tau^r} \phi(\tau' - \tau) \right)_{\tau = mT-h} + a_{mT,r}(-)^r \left(\frac{d^r}{d\tau^r} \phi(\tau' - \tau) \right)_{\tau = mT} \right\}$$
$$= \psi(\tau') = \int_0^\infty \mathcal{W}_0(\tau)\phi(\tau' - \tau)\, d\tau$$
$$(\text{for } nT - h \leq \tau \leq nT, \quad \text{all } n \geq 1) \tag{20}$$

from (19). But,

$$\int_0^\infty \mathcal{W}_0(\tau)\phi(\tau' - \tau)\, d\tau - \int_0^\infty u(-\tau)\mathcal{W}_0(\tau)\phi(\tau' - \tau)\, d\tau$$
$$= \int_0^\infty v(\tau)\mathcal{W}_0(\tau)\phi(\tau' - \tau)\, d\tau \tag{21}$$

where $v(\tau)$ is the complement of $u(-\tau)$; thus

$$1 - u(-\tau) = v(\tau) = 1, \qquad mT \leq \tau \leq mT - h$$
$$= 0, \quad \text{otherwise.}$$

Hence with the above expression, (20) becomes

$$\sum_{m=1}^{\infty} \sum_{r=0}^{D} \left\{ a_{mT-h,r}(-)^r \left(\frac{d^r}{d\tau^r} \phi(\tau' - \tau) \right)_{\tau = mT-h} + a_{mT,r}(-)^r \left(\frac{d^r}{d\tau^r} \phi(\tau' - \tau) \right)_{\tau = mT} \right\}$$
$$= \int_0^\infty v(\tau)\mathcal{W}_0(\tau)\phi(\tau' - \tau)\, d\tau,$$
$$nT - h \leq \tau' \leq nT, \quad \text{all} \quad n > 0. \tag{22}$$

Thus let $\tau' = qT + \bar{\tau}' - h$, $0 < \bar{\tau}' < h$, $q \geq 1$ and (22) takes a more manipulable form:

$$\sum_{m=1}^{\infty} \sum_{r=0}^{D} \left\{ a_{mT-h,r}(-)^r \left(\frac{d^r}{d\tau^r} \phi(qT + \bar{\tau}' - \tau - h) \right)_{\tau = mT-h} + a_{mT,r}(-)^r \left(\frac{d^r}{d\tau^r} \phi(qT + \bar{\tau}' - \tau - h) \right)_{\tau = mT} \right\}$$
$$= \sum_{m=0}^{\infty} \int_0^{T-h} \mathcal{W}_0(mT + \tau')\phi((q - 1 - m)T$$
$$+ \bar{\tau}' - \tau' + T - h)\, d\tau'. \quad 0 \leq \bar{\tau}' \leq h \tag{23}$$

The right side of (23) may be given interpretation as a sum of decay terms for a fictitious system with various delays and of impulse response $\phi(\tau)$; the "driving function" $\mathcal{W}_0(\tau)$ is essentially switched off at the times of observation, $qT + \bar{\tau}' - h$, $0 \leq \bar{\tau}' \leq h$, and only the poles of $\phi(\tau)$ are really determining the continuous behavior of the output response. The set of coefficients and derivatives of the left side are also, effectively, decay terms depending on $\phi(\tau)$.

Now, as mentioned previously [see (10)], $\phi(\tau)$ is assumed a series of exponentials,

$$\phi(\tau) = \sum_{k=0}^{D} C_K \phi_K(\tau), \qquad \phi_k(\tau) = e^{-\alpha_K |\tau|}. \tag{24}$$

Thus,

$$\phi_K(\tau) = \phi_K^+(\tau) + \phi_K^-(\tau) \tag{25}$$

where the two components ϕ_K^+ and ϕ_K^- are zero for negative and positive argument, respectively. It also follows that

$$\phi_K(\tau + \tau') = \phi_K(\tau)\phi_K(\tau'); \quad \text{both} \quad \tau, \tau' > 0. \tag{26}$$

Further, $\mathcal{W}_0(t)$, the solution for the non-pulse-modulated case, may generally be expressed as a sum,

$$\mathcal{W}_0(\tau) = \sum_K w_K(\tau),$$

for $w_K(\tau)$ such that

$$\mathcal{W}_0(\tau + \tau') = \sum_K \bar{w}_K(\tau')w_K(\tau); \quad \text{both} \quad \tau', \tau > 0; \tag{27}$$

that is, $\mathcal{W}_0(\tau)$ is a vector in $\mathfrak{I}_\tau$. The sequences $a_{mT-h,r}$ and $a_{mT,r}$, by virtue of their reference to initial and terminal points of each pulsing interval of $\mu(-\tau)$ for positive τ, [from (18) since $\mathcal{W}$ is realizable], necessarily must be physically-realizable digital-filtering processes, *i.e.*,

$$a_{mT-h,r} = 0, \qquad m < 1$$

$$a_{mT,r} = 0, \qquad m < 0. \tag{28}$$

Substitution of (24) through (28) into (23) and consideration of the positiveness of $T - h$ and $\bar{\tau}'$ [in reference to (26) and (27)] obtains

$$\Bigg[\sum_{m=1}^{q} \sum_K \Bigg\{ \sum_r \{a_{mT-h,r} + a_{(m-1)T,r}\phi_K(T-h)\}(-\alpha_K)^r - \sum_{K'} w_{K'}((m-1)T) \int_0^{T-h} \bar{w}_{K'}(\tau)\phi_K^{-1}(\tau)\, d\tau \Bigg\} \cdot \phi_K^{+}((q-m)T)\phi_K^{+}(\bar{\tau}') \Bigg]$$
$$+ \Bigg[\sum_{m=q+1}^{\infty} \sum_K \Bigg\{ \sum_r \{a_{mT-h,r} + a_{(m-1)T,r}\phi_K^{-1}(T-h)\}\alpha_K^r - \sum_{K'} w_{K'}((m-1)T) \int_0^{T-h} \bar{w}_{K'}(\tau)\phi_K(\tau)\, d\tau \Bigg\} \cdot \phi_K^{-}((q-m)T)\phi_K^{-}(\bar{\tau}') \Bigg] = 0$$
$$\text{all} \quad q \geq 1. \tag{29}$$

In order for the above to hold for all $q \geq 1$, for a time-invariant process, terms corresponding to the $+$, $-$ superscripts must be equated to zero separately. Thus each of the square-bracketed terms must equal zero. For such generality, it is then further necessary that all terms with a given m be equated to zero. This produces a pair of equations for each m.

$$\sum_K \Bigg\{ \sum_r \{a_{mT-h,r} + a_{(m-1)T,r}\phi_K^{\pm}(T-h)\}(\mp\alpha_K)^r - \sum_{K'} w_{K'}((m-1)T) \int_0^{T-h} \bar{w}_{K'}(\tau)\phi_K^{\mp}(\tau)\, d\tau \Bigg\} \cdot \phi_K^{\pm}(\bar{\tau}')\phi_K^{\pm}((q-m)T) = 0.$$
$$\text{for all} \quad q, m \geq 1 \tag{30}$$

[the pair of equations implied by the superscripts $(\pm)$]. Now, since the limits of the r subscript terms are from 0 to D and there are also $D + 1$ of the K's, this pair of equations, (30), may be separated into a system of $D + 1$ pairs, one pair for each K. Thus we have the simplified system, which for each given m is independent of q.

$$\sum_r \{a_{mT-h,r} + a_{(m-1)T,r}\phi_K(T-h)\}(-\alpha_K)^r = \sum_{K'} w_{K'}((m-1)T) \int_0^{T-h} \bar{w}_{K'}(\tau)\phi_K^{-1}(\tau)\, d\tau$$

$$\sum_r \{a_{mT-h,r} + a_{(m-1)T,r}\phi_K^{-1}(T-h)\}(\alpha_K)^r = \sum_{K'} w_{K'}((m-1)T) \int_0^{T-h} \bar{w}_{K'}(\tau)\phi_K(\tau)\, d\tau.$$
$$K = 0, 1, \cdots, D \quad \text{and} \quad m \geq 1 \tag{31}$$

The author has previously obtained the above system of equations by recourse to the z transform [4] over the time points mT for the various sequences concerned in (29). However, it is felt that the above discussion (actually of the implications of time invariance), is as sufficient and more condensed.

The coefficient matrix for the $a_{mT-h,r}$ and $a_{mT,r}$'s is generally nonsingular, which is to be expected for distinct α_K''s.

Thus, the evaluation of the δ-function derivative term coefficients is reduced to solving of a simplified, consistent set of linear equations (31). With such a solution set, (9) is formally solved but $\psi(\tau)$ in the nonstationary case is a function of N^2 Lagrangian multipliers $\lambda_{KK'}$ [from (6)]. A substitution of the solution to (29) into the N^2 equations of constraint (4) should then determine the multipliers uniquely for the final explicit solution to (5) for $\bar{W}(\tau)$. Then from (3) the actual optimal weighting function becomes

$$W(\tau) = \frac{1}{g'(-\tau)} \bar{W}(\tau)$$

where $g'(\tau)$ is equal to $g(\tau)$, the periodic pulse modulator, over the nonzero intervals and is continuous in between [discussion before (3)].

Part III—Finite Memory Considerations

In Parts I and II, the infinite memory problem was solved. It was shown that the general solution was the conventional Wiener Solution for the unmodulated case, plus a series of appropriately weighted delta-function-derivative terms at each end point of the pulsing intervals.

Finite Memory Considerations

The preceding parts of this study have been concerned with infinite memory filtering. That is, the linear operation is performed on the entire past of the input time series. Formally this implies an indefinite delay before the optimal output is generated as a time-ordered sequence. Practically, this imposes a delay commensurate with an effective over-all system decay constant. In cases where the optimum response is necessary only at some preassigned time point, and where the delay time is more restricted, the problem takes the form of a finite memory optimization. Such an optimization of unmodulated time series has been carried out in reference [2]. The following is offered as a means for utilizing and extending the analysis for unmodulated inputs to apply to the periodic pulse-modulation case.

The transition to the finite memory problem is made by replacing ∞ as a limit in all integrals by PT (except where an ideal operation is denoted). This limits the dura-

n of operation of the filter to an integral multiple, P, the pulsing period, T; so PT is the memory extent of e filter. Thus in the subsequent analysis, finite memory uations will usually have infinite memory counterparts d, for the sake of convenience, use will be made of the rresponding infinite memory equation number *primed.* r example, the infinite memory case

$$-\tau')\left\{\int_0^\infty d\tau \mathcal{W}(\tau)\phi(\tau'-\tau)-\psi(\tau')\right\}=0, \quad \tau'>0 \qquad (9)$$

here $\mathcal{W}(\tau) = 0$, for $\tau < 0$ and $mT \leq \tau \leq mT - h$, all tegers m, and the finite case becomes

$$-\tau')\left\{\int_0^{PT} d\tau \mathcal{W}(\tau)\phi(\tau'-\tau)-\psi(\tau')\right\}$$

$$= 0, \qquad PT > \tau' > 0 \qquad (9)'$$

here $\mathcal{W}(\tau) = 0$ for $PT \leq \tau \leq 0$ and $mT \leq \tau \leq mT - h$, integers, m.

The mode of solution in the infinite memory case has en first to solve for the unpulsed optimum filter, then showing that the filter for the pulsed input (with the ndition of sampled output) effectively reduces to a lsed-memory device (8) to determine the coefficients of e $D + 1$ derivatives of δ-functions at the end points of ch pulse.

The same method is to be used in the finite memory oblem. The optimum filter for the unpulsed input in is case is first formally obtained (for instance by the ethod in reference [3]). Then (9)′ requires $D + 1$ derivaves of δ-functions at each end of the pulsing intervals, it here there are only P pulses instead of infinitely many. hus (8) indicates a finite pulsed-memory filter and the llowing seems feasible: given $\mathcal{W}_0'(\tau)$ such that

$$\int^{PT} d\mathcal{W}_0'(\tau)\phi(\tau'-\tau)-\psi(\tau')=0, \quad 0\leq\tau'\leq PT \qquad (19)'$$

en by utilizing $u(-\tau)\mathcal{W}_0'(\tau)$ as the part of the weighting nction $\mathcal{W}(\tau)$, which is continuous over the pulse duration d including the $D + 1$ δ-function derivatives at the dpoints of the P pulse intervals, (20)′ becomes

$$\int^{PT} d\tau u(-\tau)\mathcal{W}_0'(\tau)\phi(\tau'-\tau)$$

$$+\sum_{m=1}^{P}\sum_{r=0}^{D}\left\{a_{mT-h,r}(-)^r\left(\frac{d^r}{d\tau^r}\phi(\tau'-\tau)\right)_{\tau=mT-h}\right.$$

$$\left.+a_{mT,r}(-)^r\left(\frac{d^r}{d\tau^r}\phi(\tau'-\tau)\right)_{\tau=mT}\right\}$$

$$= \psi(\tau') = \int_0^{PT} d\tau\mathcal{W}_0'(\tau)\phi(\tau'-\tau).$$

$$nT - h \leq \tau' \leq nT,$$

$$1 \leq n \leq P, \qquad (20)'$$

Then (21)′ through (31)′ follow as in the infinite memory case.

$$\sum_r \{a_{mT-h,r} + a_{(m-1)T,r}\phi_K(T-h)\}(-\alpha_K)^r$$

$$= \sum_{K'} w_K'((m-1)T)\int_0^{T-h}\bar{w}_{K'}'(\tau)\phi_K^{-1}(\tau)\,d\tau$$

$$\sum_r \{a_{mT-h,r} + a_{(m-1)T,r}\phi_K^{-1}(T-h)\}\alpha_K^r$$

$$= \sum_{K'} w_{K'}'((m-1)T)\int_0^{T-h}\bar{w}_{K'}'(\tau)\phi_K(\tau)\,d\tau.$$

$$K = 0, 1, \cdots, D \quad \text{and} \quad m \geq 1 \qquad (31)'$$

Example

In this case we consider a purely stationary input consisting of random signal plus noise. The desired operation is again prediction. Further, let the modulator of the input time series be a uniformly-pulsed function of the form

$$g(\tau) = 1 + a\sin\frac{2\pi\tau}{T}, \quad nT \leq \tau \leq nT + h. \quad |a| < 1 \qquad (32)$$

The signal is assumed as follows, uncorrelated with the noise:

$$\phi_m(\tau) = \sigma_m^2 e^{-\beta|\tau|}, \qquad \phi_n(\tau) = \sigma_n^2 e^{-\gamma|\tau|}$$
$$\phi_{11}(\tau) = \sigma_m^2 e^{-\beta|\tau|} + \sigma_n^2 e^{-\gamma|\tau|} \qquad (33)$$

where ϕ_m and ϕ_n refer to signal and noise autocorrelation functions respectively and ϕ_{11} is the total input autocorrelation function.

The transform of the corresponding Wiener-Hopf integral equation for the unmodulated case (12) becomes

$$Y(p)\left(\frac{-k_1(p^2-k_2^2)}{(p^2-\beta^2)(p^2-\gamma^2)}\right)-\frac{Ce^{pd}}{p^2-\beta^2}=G(p). \qquad (34)$$

e^{pd} is the transform of the ideal predictor, where

$$k_1 = 2(\beta\sigma_m^2+\gamma\sigma_n^2), \quad k_2 = \sqrt{\frac{1}{k_1}(\gamma^2\beta\sigma_m^2+\beta^2\gamma\sigma_n^2)}. \qquad (35)$$

$$C = \sigma_m^2$$

Thus, as in (11),

$$Y(p) = -\frac{C}{k_1}\frac{(p+\beta)(p+\gamma)}{(p+k_2)}\left\{\frac{(p-\beta)(p-\gamma)}{p-k_2}\frac{e^{pd}}{(p^2-\beta^2)}\right\}_+ \qquad (36)$$

or, if we define,

$$k_3 = \frac{\beta+\gamma}{\beta+k_2} \qquad (37)$$

then

$$Y(p) = \frac{Ck_3}{k_2} e^{-\beta d}\left\{1 + \frac{\gamma - k_2}{p + k_2}\right\}, \tag{38}$$

and thus

$$\mathfrak{W}_0(\tau) = \frac{Ck_3}{k_2} e^{-\beta d}\{\delta(\tau) + (\gamma - k_2)e^{-k_2\tau}\}. \tag{39}$$

The form expressed in (27),

$$\mathfrak{W}_0(mT + \tau) = \frac{Ck_3}{k_2} e^{-\beta d}\{\delta_{mT,0}\delta(\tau) + (\gamma - k_2)e^{-k_2 mT}e^{-k_2\tau}\}. \tag{40}$$

Then let

$$k_4 = \frac{Ck_3}{k_2}, \qquad k_5 = (\gamma - k_2)k_4. \tag{41}$$

Thus

$$\begin{aligned} w_0(mT) &= k_4\delta(mT), & \bar{w}_0(\tau) &= \delta(\tau) \\ w_1(mT) &= k_5 e^{-k_2 mT}, & \bar{w}_1(\tau) &= \bar{e}^{k_2\tau}. \end{aligned} \tag{42}$$

Further, define

$$\begin{aligned} w_{kk'} &= \int_0^{T-h} \bar{w}_{k'}\phi_k^{-1}(\tau)\, d\tau, \\ \bar{w}_{kk'} &= \int_0^{T-h} \bar{w}_{k'}(\tau)\phi_k(\tau)\, d\tau. \end{aligned} \tag{43}$$

Thus, if

$$\phi_0(\tau) = e^{\beta\tau}, \qquad \phi_1(\tau) = e^{\gamma\tau}$$

we obtain

$$\begin{aligned} w_{00} &= 1, \\ w_{01} &= 1, \\ w_{10} &= \frac{1}{\beta - k_2} 1 - e^{(\beta - k_2)(T-h)} \\ w_{11} &= \frac{1}{\gamma - k_2} 1 - e^{(\gamma - k_2)(T-h)} \end{aligned}$$

$$\begin{aligned} \bar{w}_{00} &= 1 \\ \bar{w}_{01} &= 1, \\ \bar{w}_{10} &= \frac{1}{\beta + k_2}(1 - e^{(\beta + k_2)(T-h)}) \\ \bar{w}_{11} &= \frac{1}{\gamma + k_2}(1 - e^{(\gamma + k_2)(T-h)}). \end{aligned} \tag{44}$$

Thus the system (31), for the coefficients becomes

$$\begin{aligned} &a_{mT-h,0} - \alpha_0 a_{mT-h,1} + \phi_0(T-h)a_{(m-1),T,0} \\ &\qquad - \phi_0(T-h)\alpha_0 a_{(m-1)T,1} = \sum_{K'} w_{0K'} \\ &a_{mT-h,0} - \alpha_1 a_{mT-h,1} + \phi_1(T-h)a_{(m-1)T,0} \\ &\qquad - \phi_1(T-h)\alpha_1 a_{(m-1)T,1} = \sum_{K'} w_{1K'} \\ &a_{mT-h,0} + a_{mT-h,1} + \phi_0^{-1}(T-h)a_{(m-1)T,0} \\ &\qquad + \phi_0^{-1}(T-h)\alpha_0 a_{(m-1)T,1} = \sum_{K'} \bar{w}_{0k'} \\ &a_{mT-h,0} + \alpha_1 a_{mT-h,1} + \phi_1^{-1}(T-h)a_{(m-1)T,0} \\ &\qquad + \phi_1^{-1}(T-h)\alpha_1 a_{(m-1)T,1} = \sum_{K'} \bar{w}_{1K'}, \end{aligned} \tag{45}$$

which has the solution,

$$\begin{aligned} a_{(m-1)T,0} &= -\Delta^{-1}\{F_1(\alpha_0\alpha_1(\phi_0 + \phi_0^{-1}) - (\phi_1 - \phi_1^{-1})) \\ &\qquad + F_2(\alpha_0(\phi_0 + \phi_0^{-1}) - \alpha_1(\phi_1 - \phi_1^{-1}))\} \\ a_{(m-1)T,1} &= \Delta^{-1}\{F_1(\alpha_1(\phi_0 - \phi_0^{-1}) - \alpha_0(\phi_1 - \phi_1^{-1}) \\ &\qquad + F_2(\phi_0 + \phi_0^{-1} - \phi_1 - \phi_1^{-1})\} \end{aligned} \tag{46}$$

where,

$$\begin{aligned} \Delta = -\alpha_0\alpha_1((\phi_0 + \phi_0^{-1}) - (\phi_1 + \phi_1^{-1}))(\phi_0 + \phi_0^{-1} - \phi_1 - \phi_1^{-1}) \\ + (\alpha_0(\phi_0 - \phi_0^{-1}) - \alpha_1(\phi_1 - \phi_1^{-1})) \\ \cdot(\alpha_1(\phi_0 - \phi_0^{-1}) - \alpha_0(\phi_1 - \phi_1^{-1})) \end{aligned}$$

$$\begin{aligned} F_1 &= f_0 + f_2 - f_1 - f_3, \\ F_{2'} &= \alpha_1(f_0 - f_2) - \alpha_0(f_1 - f_3), \\ f_i &= \sum_{K'} w_{iK'}, \qquad i = 0, 1, 2, 3, \end{aligned} \tag{47}$$

and

$$\begin{aligned} a_{mT-h,0} &= \tfrac{1}{2}\{f_0 + f_2 - (\phi_0 + \phi_0^{-1})a_3 \\ &\qquad + \alpha_0(\phi_0 - \phi_0^{-1})a_4\} \\ a_{mT-h,1} &= \tfrac{1}{2}\{f_0 - f_2 - (\phi_0 - \phi_0^{-1})a_3 \\ &\qquad + \alpha_0(\phi_0 + \phi_0^{-1})a_4\}. \end{aligned} \tag{48}$$

Thus, the optimal weighting function, for a modulator of the form $\mu(\tau)(1 + a \sin 2\pi\tau/T)$ is expressible as

$$\begin{aligned} W(\tau) = \frac{u(-\tau)}{1 + a\sin 2\pi\tau/T}\{k_4\delta(\tau) + k_5 e^{-k_2\tau} \\ + \sum_{m=1}^{\infty}\sum_{r=0}^{1}\{(-)^r a_{mT-h,r}\delta^r(\tau - m\tau - h) \\ + (-)^r a_{mT,r}\delta^r(\tau - m\tau)\}\}. \end{aligned} \tag{49}$$

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The Second-Order Distribution of Integrated Shot Noise*

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Summary—Shot noise, represented by a series of impulses with Poisson distribution in time, and with arbitrary time-independent amplitude distribution, is sent through an RC integrator. Time-dependent statistics of the output are investigated by means of an integro-differential equation describing the statistical flow. The exact second-order probability density of the output is obtained. The time-dependent Edgeworth series for zero initial output is exhibited, and is seen to bear a simple relation to the familiar equilibrium Edgeworth series. Results are shown to reduce to those of the Fokker-Planck equation describing integrated "white" noise as the frequency with which impulses arrive becomes infinite.

An RC integrating circuit with impulse response $e^{-t/\tau}$ is acted upon by noise impulses, $y\delta(t - t')$, with Poisson arrivals in time with mean frequency of arrival $1/\tau_0$, and with time-independent amplitude probability density $A(y)$. Let x be the output of the integrator. The first-order equilibrium density of the output, $W(x)$, is well-known.[1] We derive here the second-order density, $W_2(x_1, x_2, \tau)$ by solving the time-dependent equation describing statistical flow.[2]

Consider an ensemble of systems with amplitude density $W(x, t)$. The development of $W(x, t)$ in time will be described by

$$\frac{\partial W(x, t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{xW}{\tau}\right) - \frac{1}{\tau_0} W(x, t) + \frac{1}{\tau_0}\int_{-\infty}^{\infty} W(x', t)A(x - x')\, dx'. \qquad (1)$$

This may be seen by considering the conditional density $P_2(x' \mid x, \Delta)$ for an infinitesimal time Δ. The probability of no pulses arriving in time Δ is $(1 - \Delta/\tau_0)$, in which event all system amplitudes decay to $x' e^{-\Delta/\tau}$. The probability of a pulse arriving is Δ/τ_0, in which case the systems jump from x' to x with probability $A(x - x')$. Hence, $P_2(x' \mid x, \Delta) = \delta(x - x' e^{-\Delta/\tau})(1 - \Delta/\tau_0) + \Delta/\tau_0 A(x - x')$ to first order in Δ. Now $W(x, t + \Delta) = \int_{-\infty}^{\infty} W(x', t)P_2(x' \mid x, \Delta)\, dx'$. Inserting the expression for P_2, differentiating with respect to Δ, and taking the limit as $\Delta \rightarrow 0$, we obtain (1).

Eq. (1) is readily solved by taking the Fourier transform of both sides. Letting $\phi(k, t) = \int_{-\infty}^{\infty} e^{ikx}\, W(x, t)\, dx$, we obtain

$$\frac{\partial\phi(k, t)}{\partial t} = -\frac{k}{\tau}\frac{\partial}{\partial k}\phi(k, t) + \frac{(\mathcal{A}(k) - 1)}{\tau_0}\phi(k, t), \qquad (2)$$

where $\mathcal{A}(k)$ is the Fourier transform of A: $\mathcal{A}(k) = \int_{-\infty}^{\infty} e^{iky} A(y)\, dy$. Letting $\phi(k, t) = e^{\psi(k,t)}$, we transform (2) into

$$\frac{\partial\psi}{\partial t} + \frac{k}{\tau}\frac{\partial\psi}{\partial k} = \frac{\mathcal{A}(k) - 1}{\tau_0}. \qquad (3)$$

The general solution to (3) is given by any particular solution plus the general solution to the homogeneous equation:

$$\frac{\partial\psi}{\partial t} + \frac{k}{\tau}\frac{\partial\psi}{\partial k} = 0. \qquad (4)$$

Physically we expect that an equilibrium distribution will be reached, so as a particular solution to (3) we choose $\psi_{eq}(k)$, a function independent of t. Eq. (3) can then immediately be integrated to give

$$\psi_{eq}(k) - \psi_{eq}(0) = \frac{\tau}{\tau_0}\int_0^k \frac{\mathcal{A}(k') - 1}{k'}\, dk'.$$

Since

$$1 = \int_{-\infty}^{\infty} W(x, t)\, dx = \phi(0, t) = e^{\psi(0,t)},$$

$\psi(0, t) \equiv 0$, and in particular $\psi_{eq}(0) = 0$. Hence,

$$\psi_{eq}(k) = \frac{\tau}{\tau_0}\int_0^k \frac{\mathcal{A}(k') - 1}{k'}\, dk'. \qquad (5)$$

The general solution to (4) is obtained by letting $u = \log k$, whence (4) becomes

$$\frac{\partial\psi}{\partial t} + \frac{1}{\tau}\frac{\partial\psi}{\partial u} = 0,$$

which has as its general solution an arbitrary function of $u - t/\tau = \log k - t/\tau = \log(ke^{-t/\tau})$. Thus, the general $\psi(k, t) = F(ke^{-t/\tau}) + \psi_{eq}(k)$ where F is determined by the boundary condition $\psi(k, 0) = \psi_0(k)$, $\psi_0(k)$ in turn being determined by an arbitrary initial x distribution: $W(x, 0) = W_0(x)$. Setting $t = 0$ in the general solution, we have $\psi(k, 0) = \psi_{eq}(k) + F(k)$, so that $F(k) = \psi_0(k) - \psi_{eq}(k)$. Thus, the general solution to (3) is

$$\psi(k, t) = \psi_{eq}(k) + \psi_0(ke^{-t/\tau}) - \psi_{eq}(ke^{-t/\tau}). \qquad (6)$$

Since $\phi(k, t) = e^{\psi(k,t)}$,

$$\phi(k, t) = \frac{\exp[\psi_{eq}(k)]\exp[\psi_0(ke^{-t/\tau})]}{\exp[\psi_{eq}(ke^{-t/\tau})]}.$$

That is,

$$\phi(k, t) = \frac{\phi_{eq}(k)}{\phi_{eq}(ke^{-t/\tau})}\phi_0(ke^{-t/\tau}). \qquad (7)$$

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[1] S. O. Rice, "Mathematical analysis of random noise," in N. Wax, ed., "Selected Papers on Noise and Stochastic Processes," Dover Publications, Inc., New York, p. 154, eq. (1.5-4); N. Y., 1954.

[2] J. E. Moyal, *J. Roy. Stat. Soc. B*, vol. 11, p. 190; 1949. Using the Langevin Equation, Moyal discusses the time-dependent statistics of Poisson noise in an RC integrator in the case where all pulses have the same amplitude. Thus, the results of the present discussion reduce to his in the case $A(y) = \delta(y - y_0)$.

From (5),

$$\phi(k, t) = \exp\left\{\frac{\tau}{\tau_0}\int_{ke^{-t/\tau}}^{k}\left[\frac{\mathcal{A}(k')-1}{k'}\right]dk'\right\}\phi_0(ke^{-t/\tau}).$$

If we let $k' = ke^{-u/\tau}$ this becomes

$$\phi(k, t) = g(k, t)\phi_0(ke^{-t/\tau}) \tag{8}$$

where

$$g(k, t) = \exp\left\{\frac{1}{\tau_0}\int_0^t [\mathcal{A}(ke^{-u/\tau}) - 1]\, du\right\}. \tag{9}$$

We now examine our solution in coordinate space. We obtain $P_2(x_0 \mid x, t)$ by letting $W_0(x) = \delta(x - x_0)$. Then $\phi_0(k) = e^{ikx_0}$ so that $\phi_0(ke^{-t/\tau})$ is the Fourier transform of $\delta(x - x_0\, e^{-t/\tau})$. If $G(x, t)$ is the Fourier transform of $g(k, t)$, $G(x, t) = 1/2\pi \int_{-\infty}^{\infty} e^{-ikx}\, g(k, t)\, dk$, then we infer from (8) that $P_2(x_0 \mid x, t)$, the transform of $\phi(k, t)$, is the convolution of $G(x, t)$ and $\delta(x - x_0e^{-t/\tau})$; *i.e.*,

$$P_2(x_0 \mid x, t) = G(x - x_0e^{-t/\tau}, t). \tag{10}$$

Given an arbitrary initial density $W_0(x)$,

$$W(x, t) = \int_{-\infty}^{\infty} W_0(x_0)P_2(x_0 \mid x, t)\, dx_0$$

so that

$$W(x, t) = \int_{-\infty}^{\infty} W_0(x_0)G(x - x_0e^{-t/\tau}, t)\, dx_0. \tag{11}$$

As $t \to \infty$, we see from (11) that $W(x, t)$ approaches a limit independent of the initial $W_0(x_0)$; *i.e.*, there is a unique equilibrium density,

$$W_{eq}(x) = G(x, \infty). \tag{12}$$

We display $W_{eq}(x)$ explicitly in terms of A by taking the Fourier transform of (9) and expressing $\mathcal{A}(k)$ in terms of its Fourier transform $A(x)$:

$$W_{eq}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ikx}$$
$$\cdot \exp\left\{\frac{1}{\tau_0}\int_0^{\infty} du \int_{-\infty}^{\infty} dy A(y)[\exp(ikye^{-u/\tau}) - 1]\right\}. \tag{12a}$$

This is in agreement with the known result for $W_{eq}(x)$.[1]

The second-order density is now given by $W_{eq}(x_0)$ $P_2(x_0 \mid x, t)$; *i.e.*,

$$W(x_0, x, t) = W_{eq}(x_0)G(x - x_0e^{-t/\tau}, t). \tag{13}$$

In addition to the existence of a unique $W_{eq}(x)$, we can verify some other properties of the solution that one would expect to hold. That $P_2(x_0 \mid x, t)$ is always positive follows from the fact that it is a solution to (1) which is initially positive, for let x' be the earliest point at which P_2 becomes negative, and let $P_2(x_0 \mid x', t)$ pass through zero at $t = t'$. Then $P_2(x_0 \mid x', t') = 0$, whereas $P_2(x_0 \mid x, t') \geq 0$ for x in the neighborhood of x', and hence

$$\frac{\partial}{\partial x}P_2(x_0 \mid x, t')\bigg|_{x=x'} = 0, \quad \frac{\partial^2}{\partial x^2}P_2(x_0 \mid x, t')\bigg|_{x=x'} \geq 0.$$

These facts, however, along with (1) require

$$\frac{\partial}{\partial t}P_2(x_0 \mid x', t)\bigg|_{t=t'} \geq 0,$$

which contradicts the assumption that P_2 becomes negative at x' an instant after t'.

For the solution to describe a Markoff process, we must have $P_2(x_0 \mid x, t) = \int_{-\infty}^{\infty} dx'\, P_2(x_0 \mid x', s)\, P_2(x' \mid x, t - s)$. Taking the Fourier transform of both sides and noting (10) and (9), we find this condition equivalent to $g(k, t) = g(k, t - s)g(ke^{-(t-s)/\tau}, s)$, which is in turn readily verified from (9).

We now obtain the time-dependent generalization of the equilibrium Edgeworth series,[3] given that systems have amplitude 0 at $t = 0$. In this case, $W(x, t) = P_2(0 \mid x, t) = G(x, t)$. The semi-invariants, λ_n, of $G(x, t)$ are defined in terms of its Fourier transform $g(k, t)$ by

$$g(k) = \exp\left\{\sum_n \frac{(ik)^n\lambda_n}{n!}\right\}.$$

Since

$$\mathcal{A}(k) = \sum_{n=0}^{\infty} i^n \frac{\overline{y^n}k^n}{n!} \quad \text{where} \quad \overline{y^n} = \int_{-\infty}^{\infty} y^n A(y)\, dy,$$

(9) may be written

$$g(k, t) = \exp\left\{\frac{\tau}{\tau_0}\sum_{n=1}^{\infty}\frac{k^n}{nn!}i^n\overline{y^n}(1 - e^{-nt/\tau})\right\}$$

from which we conclude

$$\lambda_n = \frac{\tau}{\tau_0}\frac{\overline{y^n}}{n}(1 - e^{-nt/\tau}) = \lambda_n(\infty)(1 - e^{-nt/\tau}), \tag{14}$$

where $\lambda_n(\infty)$ are the semi-invariants of the equilibrium distribution.

If we are interested in the distribution about the average $[\bar{x} = \lambda_1(t) = (\tau/\tau_0)\,\bar{y}(1 - e^{-t/\tau})]$ normalized to unit mean square at $t = \infty$, we must investigate the characteristic function $g_N(k)$ of the distribution in

$$Z = \frac{x - \bar{x}(t)}{\sigma_\infty}\left(\sigma_\infty^2 = \lambda_2(\infty) = \frac{\tau}{\tau_0}\frac{\overline{y^2}}{2}\right).$$

A simple transformation gives

$$g_N(k, t) = \exp\left\{-\frac{k^2}{2}(1 - e^{-2t/\tau}) + \sum_{n=3}^{\infty}(ik)^n\frac{\lambda_n'(t)}{n!}\right\}$$

where the semi-invariants, λ_n', of the normalized distribution are given by

$$\lambda_1' = 0$$

$$\lambda_2' = 1 - e^{-2t/\tau}$$

$$\lambda_n' = \left(\frac{\tau_0}{\tau}\right)^{(n/2)-1}\frac{2^{n/2}}{n}\frac{\overline{y^n}}{(\overline{y^2})^{n/2}}(1 - e^{-nt/\tau}) \tag{15}$$
$$= \lambda_n'(\infty)(1 - e^{-nt/\tau}).$$

[3] Rice, *op. cit.*, p. 157.

we collect terms of the same order in (τ_0/τ), we obtain e time-dependent Edgeworth series for $W(Z, t)$,

$$(Z, t) = \left\{1 - \frac{\lambda_3'}{3!}\frac{d^3}{dZ^3} + \left[\frac{\lambda_4'}{4!}\frac{d^4}{dZ^4} + \frac{(\lambda_3')^2}{72}\frac{d^6}{dZ^6}\right] + \cdots\right\}\frac{\exp\left\{-\frac{Z^2}{2(1-e^{-2t/\tau})}\right\}}{\sqrt{2\pi}\sqrt{1-e^{-2t/\tau}}}. \quad (16)$$

The λ_n' are just the coefficients of the well-known uilibrium Edgeworth series with time-dependent tors $(1 - e^{-nt/\tau})$, and the equilibrium function

$$\frac{e^{-Z^2/2}}{\sqrt{2\pi}}$$

is here replaced by

$$\frac{e^{-Z^2/2\lambda_2' t}}{\sqrt{2\pi\lambda_2'(t)}}.$$

Note that as $\tau_0 \to 0$, (16) goes into the Gaussian solution of the Fokker-Planck equation for integrated white noise. It is also readily verified from (13), (12), and (16) that the appropriately-normalized joint probability density $W(x_0, x, t)$ becomes bivariate Gaussian as $\tau_0 \to 0$. For the conclusion of (16), that the normalized $G(x, t)$ becomes gaussian as $\tau_0 \to 0$ can immediately be generalized to $G(x - x_0e^{-t/\tau}, t)$. Since $W(x_0, x, t) = G(x_0, \infty)\, G(x - x_0 e^{-t/\tau}, t)$, the Gaussian character of W in the limit $\tau_0 \to 0$ follows.

A Theorem on Cross Correlation Between Noisy Channels*

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Summary—**Two channels carry noise waveforms, $N_0(t) + N_1(t)$ d $N_0(t) + N_2(t)$, where $N_0(t)$ is a common narrow-band Gaussian se and $N_1(t)$ and $N_2(t)$ are independent narrow-band Gaussian ses associated with each channel. The outputs of each channel sent through detectors whose outputs, $F(x, y)$, are identical nogeneous functions of the components, x and y, of their inputs where $N(t) = x(t) \cos \omega_0 t + y(t) \sin \omega_0 t$. Let $R_{12}(\tau)$ be the rmalized cross-correlation function of the two detector outputs. s shown that to determine $R_{12}(\tau)$ it suffices to know the normal-d auto-correlation function $R_0(\tau)$ of the output of a single such ector when the input is $N_0(t)$; *i.e.*, if $R_0(\tau) = G(\sigma_0^2, \rho(\tau))$ where) and σ_0 are the normalized auto-correlation function and rms either component of N_0, then it is shown that[1] $R_{12}(\tau) = G(\sigma_0^2, \tau))$ where $Z = [\{1 + (\sigma_1^2/\sigma_0^2)\}\{(1 + (\sigma_2^2/\sigma_0^2)\}]^{-\frac{1}{2}}$.**

LET narrow-band Gaussian noise be represented by $N(t) = x(t) \cos \omega_0 t + y(t) \sin \omega_0 t$, where $x(t)$ and $y(t)$ are independent Gaussian noise processes with ntical distributions. We say that a detector is homo-neous if its output at time t, F_t, is a homogeneous ction of order ν of the components of its narrow-band ise input; *i.e.*,

$$F_t = F(x(t), y(t)) \quad (1)$$

ere

$$F(\lambda x, \lambda y) = \lambda^\nu F(x, y) \quad \text{for all} \quad \lambda. \quad (2)$$

Such a detector might be, for example, a square law detector, with $F = x^2 + y^2 (\nu = 2)$; an envelope detector, with $F = \sqrt{x^2 + y^2}$ $(\nu = 1)$; or a phase detector, with $F = x/\sqrt{x^2 + y^2}$ $(\nu = 0)$.

The normalized auto-correlation function of F will depend only on the rms of either component, $\sigma = \sqrt{\overline{x^2}} = \sqrt{\overline{y^2}}$, and the normalized auto-correlation function of either component:

$$\rho(\tau) = \overline{x(t)x(t+\tau)}/\sigma^2 = \overline{y(t)y(t+\tau)}/\sigma^2,$$

I.e.,

$$R(\tau) = \frac{\overline{F_tF_{t+\tau}} - \overline{F}^2}{\overline{F^2} - \overline{F}^2} = G(\sigma^2, \rho(\tau)). \quad (3)$$

Consider now two channels having identical homogeneous detectors with noise inputs $N_0 + N_1$ in channel 1 and $N_0 + N_2$ in channel 2, N_1 and N_2 being independent noise waveforms associated with channels 1 and 2 respectively, and N_0, a third independent noise waveform common to both channels. N_0, N_1, and N_2 are all narrow-band Gaussian noise. $N_i(t) = x_i(t) \cos \omega_0 t + y_i(t) \sin \omega_0 t$, $i = 0, 1, 2$. Let $(F_1)_t$ be the output of the first detector,

$$(F_1)_t = F(x_0(t) + x_1(t), y_0(t) + y_1(t)). \quad (4)$$

Similarly, the output of the second detector is

$$(F_2)_t = F(x_0(t) + x_2(t), y_0(t) + y_2(t)). \quad (5)$$

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1 Several previous papers discuss the cross-correlation function Gaussian processes after nonlinear no-memory transformation.

It is sometimes desirable to compare the two channels by cross-correlating the normalized detector outputs rather than the narrow-band signals themselves. The normalized cross-correlation function of the output of the two detectors is

$$R_{12}(\tau) = \frac{\overline{(F_1)_t(F_2)_{t+\tau}} - \overline{F_1}\,\overline{F_2}}{\sqrt{(\overline{F_1^2} - \overline{F_1}^2)(\overline{F_2^2} - \overline{F_2}^2)}}. \tag{6}$$

It is our purpose to show that if the normalized auto-correlation function of a single such device is known as a function, $G(\sigma^2, \rho(\tau))$, of the normalized auto-correlation function of the input noise components, then R_{12} may be obtained simply in terms of G, *i.e.*, the following theorem holds.

For homogeneous F,

$$R_{12}(\tau) = G(\sigma_0^2, Z\rho(\tau)), \tag{7}$$

where ρ is the normalized auto-correlation function of either component of N_0; $G(\sigma_0^2, \rho)$ is the normalized auto-correlation function of a single homogeneous device F with input N_0, as defined in (3).

And

$$Z = \frac{1}{\sqrt{\left(1 + \frac{\sigma_1^2}{\sigma_0^2}\right)\left(1 + \frac{\sigma_2^2}{\sigma_0^2}\right)}} \tag{8}$$

where

$$\sigma_i^2 = \overline{x_i^2} = \overline{y_i^2}, \qquad i = 0, 1, 2.$$

To prove the theorem we use two well-known properties of the Gaussian distribution:

1) An n dimensional Gaussian distribution (with zero mean), $W(X_1 \cdots X_n)$, is uniquely determined by its $\frac{1}{2}n(n+1)$ second moments, $\overline{X_iX_j}$.
2) The sum of two Gaussian variables is itself a Gaussian variable.

Because of 1), if $X_1 \cdots X_4$ are any four Gaussian variables, then

$$\begin{aligned}\overline{F(X_1, X_2)F(X_3, X_4)} &= \int_{-\infty}^{\infty} \cdots \int F(X_1, X_2)F(X_3, X_4) \\ &\qquad \cdot W(X_1 \cdots X_4)\, dX_1 \cdots dX_4 \\ &= g_{FF}(\overline{X_1^2}, \overline{X_2^2}, \overline{X_3^2}, \overline{X_4^2}, \\ &\qquad \overline{X_1X_2}, \overline{X_1X_3}, \overline{X_1X_4}, \overline{X_2X_3}, \overline{X_2X_4}, \overline{X_3X_4}).\end{aligned} \tag{9}$$

where g_{FF} is a function of the second moments completely determined by F.

$$\text{If } X_1 = x_0 = x_0(t), \qquad X_2 = y_0 = y_0(t),$$
$$X_3 = x_0' = x_0(t+\tau), \qquad X_4 = y_0' = y_0(t+\tau)$$

then

$$\overline{X_1^2} = \overline{X_2^2} = \overline{X_3^2} = \overline{X_4^2} = \sigma_0^2, \qquad \overline{X_1X_3} = \overline{X_2X_4} = \sigma_0^2\rho(\tau),$$

and all other moments are zero. Hence from (9)

$$\begin{aligned}\overline{F_tF_{t+\tau}} &= \overline{F(x_0, y_0)F(x_0', y_0')} \\ &= g_{FF}(\sigma_0^2, \sigma_0^2, \sigma_0^2, \sigma_0^2, 0, \sigma_0^2\rho(\tau), 0, 0, \sigma_0^2\rho(\tau), 0) \\ &= g_{(1)}(\sigma_0^2, \rho(\tau)).\end{aligned} \tag{10}$$

Now let

$$X_1 = \alpha_1(x_0 + x_1), \qquad X_2 = \alpha_1(y_0 + y_1),$$
$$X_3 = \alpha_2(x_0' + x_2), \qquad X_4 = \alpha_2(y_0' + y_2)$$

where

$$\alpha_i = [1 + (\sigma_i^2/\sigma_0^2)]^{-1/2}, \qquad i = 1, 2.$$

Because of 2), $X_1 \cdots X_4$ are Gaussian variables so that (9) may be used. The moments are

$$\overline{X_1^2} = \overline{X_2^2} = \alpha_1^2(\sigma_0^2 + \sigma_1^2) = \sigma_0^2,$$
$$\overline{X_3^2} = \overline{X_4^2} = \alpha_2^2(\sigma_0^2 + \sigma_2^2) = \sigma_0^2,$$
$$\overline{X_1X_3} = \overline{X_2X_4} = \alpha_1\alpha_2\rho\sigma_0^2 = Z\rho\sigma_0^2,$$

and all other moments are zero. Hence

$$\begin{aligned}&\overline{F(\alpha_1(x_0 + x_1), \alpha_1(y_0 + y_1))F(\alpha_2(x_0' + x_2), \alpha_2(y_0' + y_2))} \\ &\quad = g_{FF}(\sigma_0^2, \sigma_0^2, \sigma_0^2, \sigma_0^2, 0, Z\rho\sigma_0^2, 0, 0, Z\rho\sigma_0^2, 0) \\ &\quad = g_{(1)}(\sigma_0^2, Z\rho(\tau)).\end{aligned} \tag{11}$$

Due to the homogeneity of F, the left side of (11) is equal to

$$\alpha_1^\nu\alpha_2^\nu\overline{F(x_0 + x_1, y_0 + y_1)F(x_0' + x_2, y_0' + Y_2)} = Z^\nu\,\overline{(F_1)_t(F_2)_{t+\tau}}.$$

Thus

$$\overline{(F_1)_t(F_2)_{t+\tau}} = \frac{1}{Z^\nu}\, g_{(1)}(\sigma_0^2, Z\rho(\tau)). \tag{12}$$

To obtain the remaining moments that make up $R(\tau)$ and $R_{12}(\tau)$, we make use of similar considerations in two dimensions, *i.e.*, if $H(X_1, X_2)$ is a function of two Gaussian variables then

$$\int_{-\infty}^{\infty}\int H(X_1, X_2)\, dX_1\, dX_2 = g_H(\overline{X_1^2}, \overline{X_2^2}, \overline{X_1X_2}).$$

If $X_1 = x_0$, $X_2 = y_0$, then $\overline{X_1^2} = \overline{X_2^2} = \sigma_0^2$, $\overline{X_1X_2} = 0$. Hence

$$\overline{H(x_0, y_0)} = g_H(\sigma_0^2, \sigma_0^2, 0). \tag{13}$$

When $X_1 = \alpha_i(x_0 + x_i)$, $X_2 = \alpha_i(y_0 + y_i)$, $i = 1, 2$, then $\overline{X_1^2} = \overline{X_2^2} = \sigma_0^2$, $\overline{X_1X_2} = 0$ and

$$\overline{H(\alpha_i(x_0 + x_i), \alpha_i(y_0 + y_i))} = g_H(\sigma_0^2, \sigma_0^2, 0).$$

If H is homogeneous of degree μ,

$$\alpha_i^\mu\,\overline{H(x_0 + x_i, y_0 + y_i)} = g_H(\sigma_0^2, \sigma_0^2, 0), \quad i = 1, 2. \tag{14}$$

If $H \equiv F$, then $\mu = \nu$ and (13) gives

$$\overline{F} = g_F(\sigma_0^2, \sigma_0^2, 0) = g_{(2)}(\sigma_0^2). \tag{15}$$

. (14) gives

$$= \frac{1}{\alpha_i^{\nu}} g_F(\sigma_0^2, \sigma_0^2, 0) = \frac{1}{\alpha_i^{\nu}} g_{(2)}(\sigma_0^2), \quad i = 1, 2. \quad (16)$$

oreover if $H \equiv F^2$, $\mu = 2\nu$ and (13) gives

$$\overline{F^2} = g_{F^2}(\sigma_0^2, \sigma_0^2, 0) = g_{(3)}(\sigma_0^2). \quad (17)$$

. (14) gives

$$= \frac{1}{\alpha_i^{2\nu}} g_{F^2}(\sigma_0^2, \sigma_0^2, 0) = \frac{1}{\alpha_i^{2\nu}} g_{(3)}(\sigma_0^2), \quad i = 1, 2. \quad (18)$$

As a result of (10), (15), and (17), (3) can be written

$$R(\tau) = \frac{g_{(1)}(\sigma_0^2, \rho(\tau)) - (g_{(2)}(\sigma_0^2))^2}{g_{(3)}(\sigma_0^2) - (g_{(2)}(\sigma_0^2))^2}. \quad (19)$$

On the other hand, placing the results of (12), (16), and 8) in (6), we find that factors of $1/\alpha_1^{\nu}\alpha_2^{\nu} = 1/Z^{\nu}$ appear in both numerator and denominator and cancel to give

$$R_{12}(\tau) = \frac{g_{(1)}(\sigma_0^2, Z\rho(\tau)) - (g_{(2)}(\sigma_0^2))^2}{g_{(3)}(\sigma_0^2) - (g_{(2)}(\sigma_0^2))^2}. \quad (20)$$

A comparison of (19) and (20) now reveals the validity of Theorem 7.

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Signal-to-Noise Ratios in Smooth Limiters*

JANIS GALEJS†

Summary—**Signal-to-noise ratios associated with smooth band-ss limiting and subsequent narrow-band filtering of a periodic nal and random noise are computed. Observed changes in nal-to-noise ratios may be used to estimate detectability losses. e error function is used to represent the limiter characteristic various degrees of limiting. First-order corrections with an reasing input signal to the signal-to-noise ratios, which are sed on the small signal theory, are computed for limiter input ise with sin x/x, Gaussian, and exponential correlation functions.**

Introduction

LIMITING or clipping of noise has been studied by several authors.[1-6] Hard-limiting of a signal plus noise has been investigated as a limiting case of a symmetrical n-th root circuit with n approaching infinity.[7] Such an operation degrades the output signal-to-noise ratio of a band-pass limiter by a factor of $(4/\pi)$ relative to the linear case for small input signal-to-noise ratios. The deterioration of signal detectability by similar band-pass limiters has been determined for several spectral shapes of the limiter input noise.[8] The deterioration in signal detectability ranges from approximately 6 to 16 per cent. The above results suggest that only small changes in the output signal-to-noise ratio in a narrow-band filter following the wide band-pass limiter can be expected as the limiter action is varied from a linear characteristic through various degrees of limiting to hard-limiting referred to above.[7,8] However, in systems employing a multitude of parallel band-pass filters, a small per cent difference of signal-to-noise ratio in each of the channels may result in significant changes of the over-all false alarm rate or incorrect dismissal probability. The significance of determining the exact amount of signal-to-noise deterioration by various saturation or limiting levels of the filters at different input signal-to-noise ratios is apparent in such systems applications.

The narrow-band filter which follows the limiter restores to a large extent the Gaussian character of the noise

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which is lost in the wide-band limiter outputs.[9,10] This results from the averaging action of the narrow-band filter and is a consequence of the central limit theorem.[10] The computed signal-to-noise ratio when used in conjunction with a Gaussian amplitude distribution gives a first-order solution for changes in the false alarm rates and incorrect dismissal probabilities in a subsequent threshold device. A more accurate higher-order solution is obtainable by better approximations to the actual probability distribution in the narrow-banded limiter output. Although the determination of the exact probability distribution of the narrow-band filter is a rather complex problem, approximations of various degrees to it can be worked out by applying Edgeworth series.[11,12]

The specific system considered in this paper consists of a limiter followed by a narrow-band filter, as shown in Fig. 1. The waveform $e_i(t)$ is wide-band noise plus a

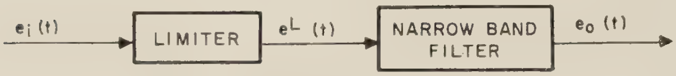

Fig. 1—Block diagram of the limiter circuit.

sinusoidal signal. The limiting function relating the limiter output to its input

$$e^L(t) = L[e_i(t)]$$

is selected such that it: 1) approximates the operational characteristics of physically realizable limiters with a linear small signal behavior and with a gradual saturation and, 2) yields readily integrable expressions in the subsequent analysis. The error function

$$Erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2)\, dx \tag{2}$$

satisfies both requirements. With

$$L(z) = \pi\, Erf\left(\frac{z}{2\sqrt{a}}\right), \tag{3}$$

the parameter 'a' determines the limiting level as indicated in Fig. 2. It is seen that for 'a' small, $L(z)$ approximates the characteristic of a hard limiter. For 'a' large, $L(z)$ acts as a linear device of gain

$$A = \lim_{z \to 0} \left(\frac{dL}{dz}\right) = \sqrt{\frac{\pi}{a}}. \tag{4}$$

[9] A. J. F. Siegert, "Passage of stationary processes through linear and non-linear devices," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 4–25; March, 1954.

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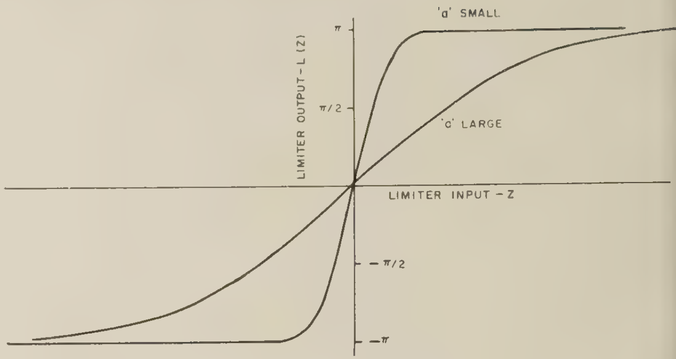

Fig. 2—Limiter characteristic.

This shape of the limiting function is the only one known that gives for a noise-only input a simple closed-form expression for the autocorrelation function of the limiter output at various degrees of limiting.[5]

Autocorrelation Function of the Limiter Output

The characteristic function method of Rice[13] is used for computing the autocorrelation function of the limiter output. The autocorrelation function of the limiter output is

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int_{-\infty}^{\infty} du\, \overline{L(u)} \exp(-0.5\sigma^2 u^2)$$
$$\cdot \int_{-\infty}^{\infty} dv\, \overline{L(v)} \exp[-0.5\sigma^2 v^2 - \sigma^2 \rho(\tau) uv]$$
$$\cdot \exp[juS(t) + jvS(t+\tau)], \tag{5}$$

where

$$L(z) = \int_{-\infty}^{\infty} \overline{L(u)} \exp(jzu)\, du \tag{6}$$

$S(t)$ is the signal part of $e_i(t)$

σ is the rms value of input noise and

$\rho(\tau)$ is its normalized autocorrelation function.

For the limiting function $L(z)$ of (3), the Fourier transform is given by[14]

$$\overline{L(u)} = \exp(-au^2)/(ju). \tag{7}$$

Assuming a sinusoidal signal and computing the time average of (5) prior to evaluating the integrals over u and v gives

[13] S. O. Rice, "Mathematical analysis of random noise," Bell Sys. Tech. J., vol. 23, pp. 282–332, 1944; and vol. 24, pp. 46–156; sec. 4.8, 1945.

[14] G. A. Campbell and R. M. Foster, "Fourier Integrals for Practical Applications," D. Van Nostrand Co., New York, N. Y.; 1948. See also, A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, "Tables of Integral Transforms," McGraw-Hill Book Co., New York, N. Y., vol. I; 1954.

This pair of Fourier transforms is depicted in Fig. 4, p. 15 and can be derived from the transform pair 725.1, p. 86 of Campbell and Foster, or from the relation (2.4.21), p. 73 of Erdelyi, *et al.*

$$\tau) = \int_{-\infty}^{\infty} du\, \overline{L(u)} \exp(-0.5\sigma^2 u^2)$$

$$\cdot \int_{-\infty}^{\infty} dv\, \overline{L(v)} \exp(-0.5\sigma^2 v^2)$$

$$\cdot \exp(-\sigma^2 \rho(\tau) uv) J_0(S\sqrt{u^2 + v^2 + 2uv \cos \omega_0 \tau}). \quad (8)$$

series expansion of the expression in the square brackets sults in[15]

$$R(\tau) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} [\sigma^2 \rho(\tau)]^k h_{nk}^2 \epsilon_n \cos n\omega_0 \tau \quad (9)$$

here

$$h_{nk} = j^{n+k} \int_{-\infty}^{\infty} \overline{L(u)}\, u^k J_n(Su) \exp(-0.5\sigma^2 u^2)\, du \quad (10)$$

$$\epsilon_n = 2 - \delta_{0n}. \quad (11)$$

ıbstituting (7) in (10), defining

$$b = a + 0.5\sigma^2 \quad (12)$$

ıd

$$w = \frac{S^2}{4b} = \frac{S^2}{4(a + 0.5\sigma^2)} = \frac{R_s(0)}{\sigma^2 + 2a}, \quad (13)$$

e evaluation of the integral gives[16]

$$_k = \begin{cases} \dfrac{\pi b^{-0.5k} w^{0.5n}}{n!\Gamma\left(\dfrac{2-k-n}{2}\right)} {}_1F_1\left(\dfrac{k+n}{2}; n+1; -w\right) & \\ & \text{for } n + k \text{ odd} \\ 0 & \text{for } n + k \text{ even} \end{cases} \quad (14)$$

hich reduces for hard limiting ($a = 0$) to the result of avenport.[7] The fundamental component of the signal ıtput is the $k = 0$, $n = 1$ term. Substituting h_{10} in (9),

$$R_s^L(\tau) = 2\left[\frac{\pi\sqrt{w}}{\Gamma(0.5)} {}_1F_1(0.5; 2; -w)\right]^2 \cos \omega_0 \tau. \quad (15)$$

he noise terms of the correlation function are all the rms with $k \neq 0$. Substituting (14) in (9)

$$R_n^L(\tau) = \sum_{\substack{n=0 \\ (n+k=odd)}}^{\infty} \sum_{k=1}^{\infty} \frac{[\sigma^2 \rho(\tau)]^k}{k!} \frac{\pi^2 b^{-k} w^n}{\left[n!\Gamma\left(\dfrac{2-k-n}{2}\right)\right]^2}$$

$$\cdot \left[{}_1F_1\left(\frac{k+n}{2}; n+1; -w\right)\right]^2 \epsilon_n \cos n\omega_0 \tau. \quad (16)$$

ith the parameter 'a' always larger than zero, w of (13) ill be less than the input signal-to-noise power ratio. or small signal-to-noise ratios, one may ignore the higher owers of w. Considering only the two lowest power terms w, (15) and (16) simplify to

$$R_s^L(\tau) \approx 2\pi w(1 - 0.5w) \cos \omega_0 \tau \quad (17)$$

and

$$R_n^L(\tau) \approx 2\pi\left(W + \frac{1}{2}\frac{W^3}{3} + \frac{1.3}{2.4}\frac{W^5}{5} + \cdots\right)$$

$$+ 2\pi w\left[-\left(W + \frac{1}{2}W^3 + \frac{1.3}{2.4}W^5 + \cdots\right)\right.$$

$$\left. + \left(\frac{1}{2}W^2 + \frac{1.3}{2.4}W^4 + \cdots\right)\cos \omega_0 \tau\right] \quad (18)$$

where

$$W = \frac{\sigma^2 \rho(\tau)}{\sigma^2 + 2a} = \frac{\sigma^2 \Phi(\tau) \cos \omega_1 \tau}{\sigma^2 + 2a}. \quad (19)$$

The first summation of (18) is recognized as the $\sin^{-1} W$ function.[5] An expression analogous to the autocorrelation function of McFadden[6] is obtained by neglecting terms containing w^3 and higher powers in (16) and by letting $a = 0$ (hard limiting).

Output of the Narrow-Band Filter

The autocorrelation function of the filter output is given by the convolution

$$R_d(\tau) = R(\tau) * \mathfrak{F}^{-1} \mid Z(\omega) \mid^2 \quad (20)$$

where $Z(\omega)$ is the transfer function of the narrow-band filter. Letting

$$R_z(\tau) = \mathfrak{F}^{-1} \mid Z(\omega) \mid^2, \quad (21)$$

the mean-square filter output is simply

$$R_d(0) = \int_{-\infty}^{\infty} R(x) R_z(x)\, dx. \quad (22)$$

A single tuned RLC filter is assumed for the narrow-band filter.[17] In a high Q narrow-band filter, its natural frequency is much larger than the filter half-power bandwidth

$$\omega_d = 2d. \quad (23)$$

The filter impulse response and also the function $R_z(\tau)$ may be approximated by [18]

$$R_z(\tau) \approx K \exp(-d \mid \tau \mid) \cos \omega_c \mid \tau \mid. \quad (24)$$

The mean-square signal output of the narrow-band filter is computed as indicated in (22) with $R_z(\tau)$ given by (24) and the autocorrelation function of the limiter signal output $R_s^L(\tau)$ given by (17). Lining up the natural frequency of the bandpass filter with the signal frequency,

$$\omega_c = \omega_0. \quad (25)$$

15 See sec. 4.9 of Rice, *op. cit.*

16 See (4.10–5) of Rice, *ibid.*, or the development of Davenport, . *cit.*

17 Although a specific filter characteristic is assumed for the subsequent analysis, the final results, that is the degradation of the filter output signal-to-noise ratio due to limiting, are independent of the filter shape or the filter-bandwidth, provided that the filter bandwidth is much less than the bandwidth of the input noise spectrum. A straightforward computation of signal energy and of the noise spectral density at the limiter output in the vicinity of the signal frequency would yield identical results.

18 The relation (24) can be derived with the aid of (448.1) and (449.1) of Campbell and Foster, *op. cit.*

Applying (25), the signal power output of the narrow-band filter becomes

$$S_d = 4\pi K w(1 - 0.5w) \int_0^\infty \cos^2 \omega_0 x \exp(-dx)\, dx = \frac{2\pi K w}{d}(1 - 0.5w). \tag{26}$$

Substituting (26) in (22), the noise output power of the narrow-band filter becomes

$$N_d = 2K \int_0^\infty \cos \omega_0 x \exp(-dx) R_n^L(x)\, dx. \tag{27}$$

The autocorrelation function of noise as given by (18) consists of three summations. Expanding the odd W powers of the first and second summations, only those terms proportional to $\cos \omega_1 \tau$ will give rise to noise components within the filter pass band and will contribute significantly to the integral (27). The amplitude of these terms is obtained from the relation

$$\cos^{2m+1} y = 2^{-2m}\left[\binom{2m+1}{m} \cos y + \binom{2m+1}{m-1} \cos 3y + \cdots\right]. \tag{28}$$

The third summation contains only even powers of W and of $\cos \omega_1 \tau$. Only the constant part and the part proportional to $\cos 2\omega_1 \tau$ of the expansion of the even-powered terms of $\cos \omega_1 \tau$ will contribute significantly to the integral (27). The amplitude of these terms is obtained from the relation

$$\cos^{2m} y = 2^{-2m}\left[\binom{2m}{m} + 2\binom{2m}{m-1} \cos 2y + \cdots\right]. \tag{29}$$

Substituting (18) in (27) and applying (28) and (29) results in [19]

$$N_d = 2\pi K \int_0^\infty dx \exp(-dx) \cdot \left\{\cos(\omega_0 - \omega_1)x \sum_{m=0}^{\infty}\left[A_m - \frac{R_s(0)}{\sigma^2 + 2a} B_m\right] \cdot [\Phi(x)]^{2m+1} + \frac{R_s(0)}{\sigma^2 + 2a} \sum_{m=1}^{\infty} [C_m \cos(\omega_0 - \omega_1)x + D_m \cos 2(\omega_0 - \omega_1)x][\Phi(x)]^{2m}\right\} \tag{30}$$

where

$$A_m = \left[\frac{(2m)!}{2^{2m}(m!)^2}\right]^2 \frac{1}{m+1}\left(\frac{\sigma^2}{\sigma^2 + 2a}\right)^{2m+1} \tag{31}$$

$$B_m = (2m + 1) A_m \tag{32}$$

$$C_m = (m + 1) A_m \frac{\sigma^2 + 2a}{\sigma^2} \quad (3$$

$$D_m = m A_m \frac{\sigma^2 + 2a}{\sigma^2}. \quad (3$$

In the above integral, $\Phi(\tau)$ represents the envelope c the normalized noise correlation function $\rho(\tau)$, *i.e.*,

$$\rho(\tau) = \Phi(\tau) \cos \omega_1 \tau. \quad (3$$

The integrals of (30) are evaluated in the Appendix fc exponential, $\sin x/x$ and Gaussian functions $\Phi(\tau)$. Afte evaluating the summations, the noise power outpu reduces to the form

$$N_d = \frac{2\pi K}{bk} \frac{\sigma^2}{\sigma^2 + 2a}\left[1 + c\frac{R_s(0)}{\sigma^2 + 2a}\right]. \quad (3$$

In the absence of signal or for very small input signal-tc noise ratios, the part of the noise proportional to R_s(C can be neglected, and c may be set equal to zero in (36)

The output signal power-to-noise power ratio is obtaine from (26) and (36) as

$$\left(\frac{S}{N}\right)_{\rm Lim} = \frac{S_d}{N_d} \simeq k \frac{\omega_n}{\omega_d} \frac{R_s(0)}{\sigma^2}\left[1 - (0.5 + c)\frac{R_s(0)}{\sigma^2 + 2a} + \cdots\right]. \quad (37$$

Eq. (37) represents the signal-to-noise ratio if noise i measured during signal presence. If noise is measured i the absence of signal, this signal-to-noise ratio is obtaine by letting $c = 0$ in (37).

The output signal-to-noise ratio depends on the shape c the input noise spectrum, on the relative position $|\omega_0 - \omega$ of the signal filter within the noise band, and on th limiting or clipping level.

The limiting level is defined as the limiter input ampli tude for which the limiter output reaches 90 per cent c its maximum. Introducing a proportionality constant between this limiting level L and the rms value of th limiter input noise σ,

$$L = q\sigma. \quad (3$$

It follows from the limiter characteristic (3) that

$$L = 3.3\sqrt{a}. \quad (3$$

Designating the value of the constant k for a linea system (a and q infinite) by k_0,

$$k = k_0 k_1. \quad (4$$

Solving (38) and (39) for 'a', substituting the expressio for 'a' in (37) and introducing the notation

$$D = -\frac{0.5 + c}{(1 + 0.184q^2)k_0}, \quad (4$$

[19] The A_m terms of (30) are identical with the summation in (11) of Manasse, *et al.*, *op. cit.*

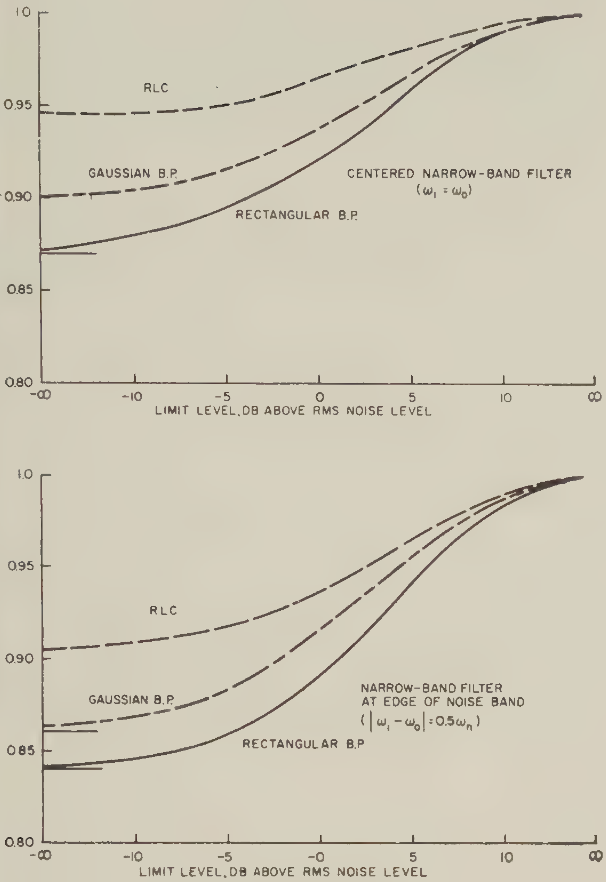

Fig. 3—Degradation constant, k_1.

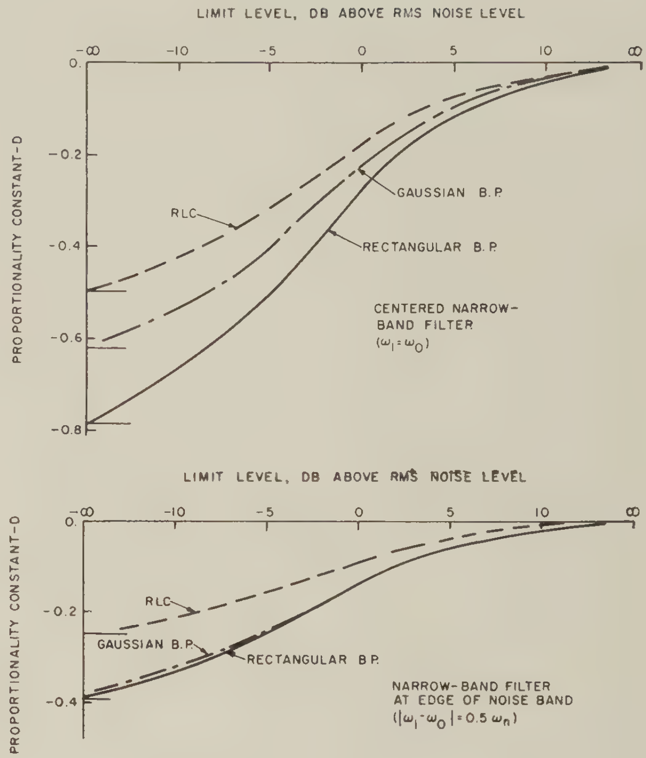

Fig. 4—First order correction to the output signal-to-noise ratio (noise measured in absence of signal).

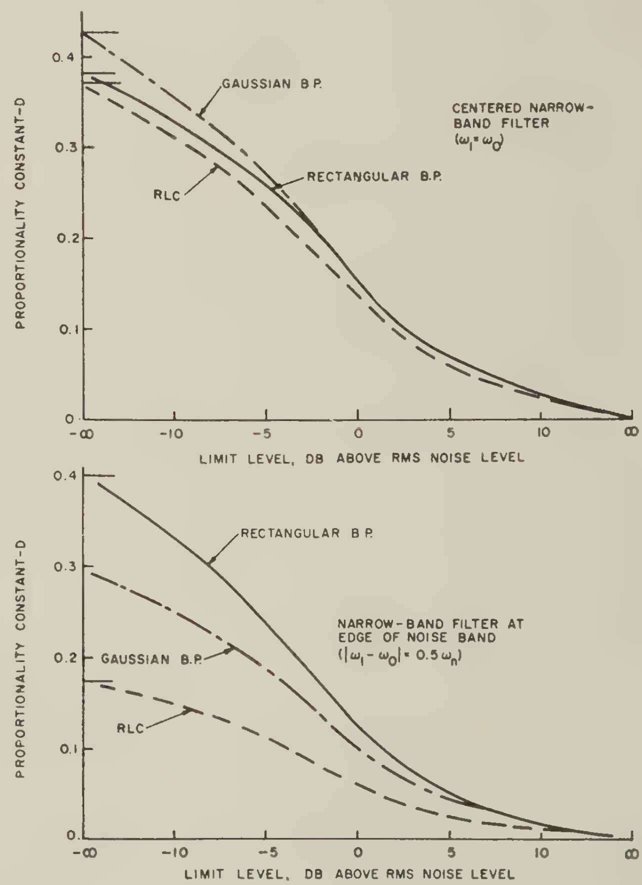

Fig. 5—First order correction to the output signal-to-noise ratio (noise measured in presence of signal).

he output signal power-to-noise power ratio becomes

$$\left(\frac{S}{N}\right)_{\text{Lim}} = \left(\frac{S}{N}\right)_{\text{No Lim}} k_1\left[1 + D\,\frac{\omega_d}{\omega_n}\left(\frac{S}{N}\right)_{\text{No Lim}} - \cdots\right] \tag{42}$$

vhere

$$\left(\frac{S}{N}\right)_{\text{No Lim}} = k_0\,\frac{\omega_n}{\omega_d}\left(\frac{S}{N}\right)_{\text{in}} \tag{43}$$

nd

$$\left(\frac{S}{N}\right)_{\text{in}} = \frac{R_s(0)}{\sigma^2}. \tag{44}$$

The plots of the constants k_1 and D are shown in Figs. 3–5 or various limiting levels q and for noise of exponential, in x/x, and Gaussian correlation functions. Noise of such orrelation functions is obtained by passing white noise hrough single tuned RLC filters, rectangular or Gaussian and-pass filters, respectively. The plots are given for $\omega_1 - \omega_0| = 0$ and $0.5\omega_n$. The values of the constant k_0 re listed in Table I. It should be noted that the input ignal-to-noise ratio of (44) was defined at the input to he limiter and that k_0 of Table I does not take into conideration the possible attenuation of the signal by skirts f the RF or IF amplifier.

TABLE I
CONSTANT k_0

	$\mid \omega_1 - \omega_0 \mid =$	0	$0.5\ \omega_n$
Correlation	Expon.	1.0	2.0
Function of	$\sin x/x$	$2/\pi$	$4/\pi$
Noise	Gaussian	$\sqrt{2/\pi}$	$1.65\ \sqrt{2/\pi}$

DISCUSSION

The output signal-to-noise ratios of a smooth limiter followed by a narrow-band filter were related to the output signal-to-noise ratios of a corresponding linear system. In the latter system, a linear amplifier is substituted for the limiter, leaving the other system parameters unaltered. The relative decrease in the output signal-to-noise ratio due to limiting, although computed for a specific filter characteristic, is numerically equal to the decrease of signal energy to noise power density ratio, which characterizes detectability of known signals in presence of white Gaussian noise. However, it should be remembered that the filter output is only approximately Gaussian. While the central limit theorem applies to the noise times noise terms of the limiter output, the signal times noise terms may yield non-Gaussian statistics no matter how narrow the output filter is made. If Gaussian statistics are used in conjunction with the computed signal-to-noise ratios to estimate losses in signal detectability, such a procedure will become less accurate with increasing signal-to-noise ratios. It is generally advisable to use the Edgeworth series approach to obtain results of higher accuracy.[11,12]

For very small input signal-to-noise ratios, the degradation of the limiter output signal-to-noise ratio is characterized by a single constant k_1.[20] This constant has numerical values within the range of 0.84 to 1.0. The output signal-to-noise ratio in the small signal case is deteriorated by no more than 16 per cent. Also, the deterioration of the output signal-to-noise ratio by the limiter action is more pronounced at the edge of the input noise band than at its center. Manasse, *et al.*,[8] observed that the signal detectability loss is less near the band edge of the rectangular input noise spectrum relative to the detectability loss near the center of the band. This conclusion cannot be generalized to hold at the band edge of the rectangular spectrum or for the other spectra considered in this paper.[21]

[20] The degradation factor of Manasse, *et al.*, *op. cit.*, is equal to the reciprocal of the constant k_1, if $\omega_d \ll \omega_n$ and $q = 0$.

[21] For a signal located near the inside edge of the rectangular input spectrum, the loss in signal detectability is less than it is for a signal located in the center of the band as long as the separation of the filter frequency ω_0 from the band edge at frequencies $(\omega_1 \pm 0.5\ \omega_n)$ exceeds the filter bandwidth ω_d. This is not the case when the filter approaches the band edge very closely, or when it is centered around the band edge where noise spectrum is discontinuous. That part of the limiter output noise spectrum which lies beyond the edges of the input noise spectrum contributes to the filter noise output. The noise output of such a filter becomes higher relative to the linear case than the output of the filter located at the center of the noise band, which implies a larger loss of detectability. This can be checked numerically by sliding a narrow band filter along the noise spectrum plotted in Fig. 3.10 of Lawson and Uhlenbeck, *op. cit.*

The signal-to-noise output ratios computed for very small input signal-to-noise ratios require corrections with increasing input signal-to-noise ratios. The first-order correction is proportional to the input signal-to-noise ratio.

The proportionality constant, D, is negative and the output signal-to-noise ratio is further degraded if noise is measured during signal absence. The absolute value of D is less than 0.8 and it is less than 0.28 for limiting levels above the rms noise level. The absolute value of D and thus the relative deterioration in the output signal-to-noise ratio with increasing input signal-to-noise ratio is more pronounced at the center of the input noise band than at its edge.

The proportionality constant, D, is positive if noise is measured during signal presence. The output signal-to-noise ratio is therefore improved relative to the small input signal case, which agrees with the conclusion of Davenport.[7] The constant D has numerical values of less than 0.5, but it is less than 0.16 for the limiting levels above the rms noise level. The constant D, and thus the relative improvement in the output signal-to-noise ratio with increasing input signal-to-noise ratio, is larger at the edge of the input noise band than at its center.

APPENDIX

NOISE POWER OUTPUT OF THE NARROW-BAND FILTER

Noise with an Exponential Autocorrelation Function

The output of a single tuned RLC filter of half-power bandwidth $\omega_n = 2b$ has an autocorrelation function of the form indicated in (35) with

$$\Phi(t) = \exp(-b \mid \tau \mid). \tag{45}$$

Substituting (45) in (30), the integrals to be evaluated are of the form

$$I_n(\omega) = \int_0^\infty \cos \omega x \exp[-(d + nb)x]\, dx$$
$$\approx \left\{ nb\left[1 + \left(\frac{\omega}{nb}\right)^2\right]\right\}^{-1} \tag{46}$$

assuming $d \ll b$.

Applying (46) to (30),

$$N_d = 2\pi K\Bigg[\sum_{m=0}^{\infty}\left[A_m - \frac{R_s(0)}{\sigma^2 + 2a}B_m\right]$$
$$\cdot\left\{(2m+1)b\left[1 + \left(\frac{\omega_0 - \omega_1}{(2m+1)b}\right)^2\right]\right\}^{-1}$$
$$+ \frac{R_s(0)}{\sigma^2 + 2a}\sum_{m=1}^{\infty} C_m\left\{2mb\left[1 + \left(\frac{\omega_0 - \omega_1}{2mb}\right)^2\right]\right\}^{-1}$$
$$+ \frac{R_s(0)}{\sigma^2 + 2a}\sum_{m=1}^{\infty} D_m\left\{2mb\left[1 + \left(\frac{2\omega_0 - 2\omega_1}{2mb}\right)^2\right]\right\}^{-1}\Bigg]. \tag{47}$$

Noise with a sin x/x Autocorrelation Function

Noise of constant spectral intensity of bandwidth $\omega_n = 2b$ has the autocorrelation of the form indicated in (35) with

$$\Phi(t) = \frac{\sin bt}{bt}. \tag{48}$$

Substituting (48) in (30), the integrals to be evaluated are of the form

$$I_n(\omega) = \int_0^\infty \exp(-dx) \cos \omega x \left(\frac{\sin bx}{bx}\right)^n dx. \tag{49}$$

Such integrals are tabulated for $n \leq 3$ only.[22] Integrals with larger values of n can be evaluated by repeated integration by parts. Restricting the consideration to the cases where $d \ll b$, the evaluation is simplified and the above integral can be approximated by

$$I_n(\omega) \approx \int_0^\infty \cos \omega x \left(\frac{\sin bx}{bx}\right)^n dx. \tag{50}$$

The per cent error is less than $(100\ d/b)$ for $n = 1$. The error is significantly less for higher values of n.

Evaluating the integral[23]

$$I_1(\omega) = \begin{cases} \dfrac{\pi}{2b} & \text{for} \quad \omega < b \\ \dfrac{\pi}{4b} & \text{for} \quad \omega = b \\ 0 & \text{for} \quad \omega > b \end{cases} \tag{51}$$

and

$$I_n(\omega) = \frac{n\pi}{b2^n} \sum_{0 \leq r < (\omega/b + n)/2} \frac{(-1)^r \left(\frac{\omega}{b} + n - 2r\right)^{n-1}}{r!(n-r)!} \tag{52}$$

with $n = 2, 3, \cdots$.

[22] D. Bierens DeHaan, "Nouvelles Tables d'Integrales Definies," Hafner Publ. Co., New York, N. Y., tables 365, 368, and 370; N. Y., 1957.

[23] See pp. 18–20 of Erdelyi, *et al.*, *op. cit.*

Using the above notation for the integrals, (30) can be rewritten as

$$N_d = 2\pi K \left\{ \sum_{m=0}^\infty \left[A_m - \frac{R_s(0)}{\sigma^2 + 2a} B_m \right] I_{2m+1}(\omega_0 - \omega_1) + \frac{R_s(0)}{\sigma^2 + 2a} \sum_{m=1}^\infty [C_m I_{2m}(\omega_0 - \omega_1) + D_m I_{2m}(2\omega_0 - 2\omega_1)] \right\}. \tag{53}$$

Noise with a Gaussian Autocorrelation Function

Noise with a Gaussian spectral intensity of bandwidth $\omega_n = 2b$ measured between the exp (-0.5) points of the power spectrum has an autocorrelation function of the form indicated in (35) with

$$\Phi(\tau) = \exp(-0.5 b^2 \tau^2). \tag{54}$$

Substituting (54) in (30), the integrals to be evaluated are of the form

$$I_n(\omega) = \int_0^\infty \cos \omega x \exp(-dx) \exp(-0.5 n b^2 x^2)\, dx. \tag{55}$$

For $d \ll b$, the above integral is readily evaluated as

$$I_n(\omega) = \frac{1}{b} \sqrt{\frac{\pi}{2n}} \exp\left[-\frac{1}{2n} \left(\frac{\omega}{b}\right)^2 \right]. \tag{56}$$

Applying (56) to (30)

$$N_d = 2\pi K b^{-1} \sqrt{0.5\pi} \left[\sum_{m=0}^\infty \left[A_m - \frac{R_s(0)}{\sigma^2 + 2a} B_m \right] \cdot \frac{1}{\sqrt{2m+1}} \exp\left[-\frac{1}{2(2m+1)} \left(\frac{\omega_0 - \omega_1}{b}\right)^2 \right] + \frac{R_s(0)}{\sigma^2 + 2a} \sum_{m=1}^\infty \left\{ \frac{C_m}{\sqrt{2m}} \exp\left[-\frac{1}{4m} \left(\frac{\omega_0 - \omega_1}{b}\right)^2 \right] + \frac{D_m}{\sqrt{2m}} \exp\left[-\frac{1}{m} \left(\frac{\omega_0 - \omega_1}{b}\right)^2 \right] \right\} \right]. \tag{57}$$

A Note on the Estimation of Signal Waveform*

DAVID MIDDLETON†

Summary—The problem of estimating signal waveform from received data that is corrupted by noise is briefly considered from the viewpoint of decision theory, in extension of some earlier work.[1] The noise is assumed to be a Gauss process, which may or may not be stationary. Here, however, nothing is known about the signal process except that it may be deterministic, entirely random, or a mixed process. Two new features in the present application are the representation of the signal process as a linear expansion (M. S.) in complete orthonormal sets, and suitable choices of these sets. Examples involving discrete and continuous sampling on a finite interval, with various choices of *a priori* distributions of signal parameters are described, including calculations of Bayes and Minimax risks.

A CENTRAL problem in communication theory is the extraction of signals from noisy backgrounds under conditions of optimality or near optimality. A general theoretical framework for treating such problems is provided by the methods of statistical decision theory[2] as recently adapted and extended to the communication process.[1,3] The purpose of the present paper is to describe briefly some new results and modifications of existing approaches which appear useful in the operation of an adequate theory, *viz.*, 1) for optimum structure and its physical realization, 2) for system evaluation, and 3) for comparison of optimum and suboptimum systems.

Our present problem is the estimation of signal waveform S when the signal process $S(t)$ is combined in some known fashion with a noise process $N(t)$. Estimation takes place on an interval $(0, T)$ and is based on the received data $V(t) = S(t) \oplus N(t)$. For the cases considered here $\oplus$ represents simple addition, and $N(t)$ is a normal process with zero mean and covariance function $K_N(t_1, t_2) = \overline{N(t_1)N(t_2)}$, while $N(t)$ need not be stationary. Here nothing is known about $S(t)$, except that S is present in V and that S may be: a) deterministic, b) entirely random, or c) a mixed process. Two new elements of the present approach are: 1) the explicit representation of the signal as a linear expansion in a (real) complete orthonormal set, *e.g.*,

$$S(t) = \sum_{k=1}^{\infty} a_k \phi_k(t) \tag{1}$$

where $\{\phi_k\}$ are orthonormal on $(0, T)$, *i.e.*, $\int_0^T \phi_k(t)\phi_l(t)\,dt = \delta_{kl}$, $(\delta_{kk} = 1;\ \delta_{kl} = 0,\ k \neq l)$, and 2) an appropriate choice of the $\{\phi_k\}$, usually on the basis of the noise background, *e.g.*, the $\{\phi_k\}$ are solutions of the homogeneous Fredholm integral equation (of the second kind):

$$\int_0^T K_N(t, u)\phi_k(u)\,du = \lambda_k \phi_k(t), \qquad (0 \leq t \leq T), \quad k = 1, \cdots, \infty, \tag{2}$$

with λ_k the (real) eigenvalues corresponding to the (real) eigenfunctions ϕ_k. (The noise covariance K_N is also assumed to be continuous and positive definite, so that the λ_k are all positive.) The $\{a_k\}$ are thus a set of (time-independent) random parameters, whose statistical properties determine those of the signal process $S(t)$, cf. (1). Specifically, from (1) we have

$$a_k = \int_0^T S(t)\phi_k(t)\,dt. \tag{3}$$

Similarly, we have for the received-data process

$$V(t) = \sum_{k=1}^{\infty} c_k \phi_k(t), \quad \text{with} \quad c_k = \int_0^T V(t)\phi_k(t)\,dt, \tag{4}$$

and for the noise

$$N(t) = \sum_{b=1}^{\infty} b_k \phi_k(t); \qquad b_k = \int_0^T N(t)\phi_k(t)\,dt, \tag{5}$$

where, (1), (3)–(5) are, of course, stochastic integrals, defined M.S. in the usual Riemannian sense.[4] Thus, we have also $c_k = a_k \oplus b_k$. The use of orthonormal expansions like the above is, of course, not new,[5] but such devices do not appear to have been previously exploited in any general approach to signal extraction problems.[6]

Quadratic Cost Function

Let us now estimate $S(t)$ at a time t_λ, $(0 \leq t_\lambda \leq T)$, with an optimum system using the quadratic cost function (QCF)

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[1] D. Middleton and D. Van Meter, "Detection and extraction of signals in noise from the point of view of statistical decision theory," *J. Soc. Ind. Appl. Math.*, vol. 3, pp. 192–253, December, 1955; vol. 4, ch. 4, sec. 2.2, pp. 86–119, June, 1956. For a preliminary account, see also D. Van Meter and D. Middleton, "Modern statistical approaches to reception in communication theory," IRE Trans. on Information Theory, vol. IT-4, pp. 119–145; September, 1954.

[2] A. Wald, "Statistical Decision Functions," John Wiley and Sons, Inc., New York, N. Y.; 1950.

[3] D. Middleton, "Random Processes, Signals, and Noise—An Introduction to Statistical Communication Theory," in "Pure and Applied Physics," Int. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., ch. 21; in press.

[4] M. S. Bartlett, "An Introduction to Stochastic Processes," Cambridge Univ. Press, Cambridge, Eng., sec. 5.11; 1955.

[5] See, for example, W. H. Huggins, "Signal theory," IRE Trans. on Circuit Theory, vol. CT-3, pp. 210–216, December, 1956 for remarks on signal representations; U. Grenander, "Stochastic processes and statistical inference," *Arkiv. Matematik*, vol. 1, sec. 3, pp. 195–227; 1950. See Almquist and Wiksells, (Stockholm, Sweden), for the use of such expansions in statistical inference. For noise analysis, see W. B. Davenport and W. L. Root, "Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., chaps. 6, 14; 1958. Also, Middleton, *op. cit.*, ch. 8.

[6] An exception is an article by E. M. Glaser and J. H. Park, Jr., "On signal parameter estimation," IRE Trans. on Information Theory, vol. CT-4, pp. 173–174; December, 1958. These authors consider some special cases of the results summarized in their article, using (1) and the methods of Middleton and Van Meter, *op. cit.*, and Middleton, *op. cit.*

$$C(S, \gamma) = C_0[S(t_\lambda) - \gamma_\sigma^*(S_\lambda \mid \mathbf{V})]^2, \tag{6}$$

where γ_σ^* is the optimum or Bayes estimator of $S(t_\lambda)$, based on the (for the moment) discretely sampled data $\mathbf{V} = [V(t_1), \cdots, V(t_n)]$ in $(0, T)$. By definition, γ_σ^* minimizes the average risk (or cost) $R_\lambda = \mathbf{E}_{V,S}\{C(S_\lambda, \gamma)\}$.

Following the usual minimization procedure we readily find for (1) in (6) that

$$\gamma_\sigma^* = \sum_{k=1}^{\infty} \gamma_k^*(\mathbf{V})\phi_k(t_\lambda) = \sum_{k=1}^{\infty} a_k^*\phi_k(t_\lambda), \tag{7}$$

where the set of Bayes estimators $\{a_k^*\}$ of $\{a_k\}$, based on $\mathbf{V}$ in $(0, T)$, is explicitly

$$\{a_k^*\} = \frac{\int_{(\mathbf{a})} \{a_k\}\sigma(a_1, \cdots, a_k, \cdots)F_n(\mathbf{V} \mid a_1, \cdots)\, da_1\, da_2 \cdots}{\int_{(\mathbf{a})} \sigma(a_1, \cdots, a_k, \cdots)F_n(\mathbf{V} \mid a_1, \cdots)\, da_1\, da_2 \cdots} \tag{8a}$$

or more compactly, for each a_k

$$a_k^* = -i\frac{d}{d\xi_k} \log \int_{(\mathbf{a})} \sigma(\mathbf{a})F_n(\mathbf{V} \mid \mathbf{a})\, d\mathbf{a}e^{i\mathbf{a}\xi} \Big|_{\xi=0}, \tag{8b}$$

where $\mathbf{a} = [a_1, a_2, \cdots]$ is an infinite element column vector, like $\boldsymbol{\xi} = [\xi_1, \xi_2, \cdots]$. Here σ is the *a priori* distribution density (d.d.) of the $\{a_k\}$ and $F_n(\mathbf{V} \mid \mathbf{a})$ is the nth order conditional d.d. of $\mathbf{V}$, given $\mathbf{a}$, which is obtained directly from (1) and $F_n(\mathbf{V} \mid \mathbf{S})$, the corresponding nth order d.d. of $\mathbf{V}$, given $\mathbf{S}$.

Gauss Noise

Eq. (8) applies for $V = S \oplus N$ and general noise statistics. Let us specialize to the familiar case of additive normal noise. Then

$$F_n(\mathbf{V} \mid \mathbf{S}) = (2\pi)^{-n/2}(\det \mathbf{K}_N)^{-\frac{1}{2}} \cdot \exp\{-(\tilde{\mathbf{V}} - \tilde{\mathbf{S}})\mathbf{K}_N^{-1}(\mathbf{V} - \mathbf{S})/2\}. \tag{9}$$

Writing the following (row) vectors (of nth and infinite orders), with $j = 1, \cdots, n$,

$$\mathbf{S} = [\sum_k a_k\phi_k(t_j)]; \quad \mathbf{\Phi}^{(n)} = [\Phi_k^{(n)}] = [\tilde{\mathbf{V}}\mathbf{K}_N^{-1}\boldsymbol{\phi}_k^{(n)}]$$
$$\boldsymbol{\phi}_k^{(n)} = [\phi_k^{(n)}(t_j)], \tag{10a}$$

and for the infinite order square matrix $\mathbf{\Psi}^{(n)}$, setting

$$\mathbf{\Psi}^{(n)} = [\Psi_{kl}^{(n)}]; \quad \Psi_{kl}^{(n)} = \tilde{\boldsymbol{\phi}}_k^{(n)}\mathbf{K}_N^{-1}\boldsymbol{\phi}_l^{(n)} = \Psi_{lk}^{(n)}, \tag{10b}$$

we may write (8b) generally as

$$a_k^*(\mathbf{V}) = -i\frac{d}{d\xi_k} \log \int_{(\mathbf{a})} \sigma(\mathbf{a}) \cdot \exp\{\tilde{\mathbf{a}}(i\boldsymbol{\xi} + \mathbf{\Phi}^{(n)}) - \tfrac{1}{2}\tilde{\mathbf{a}}\mathbf{\Psi}^{(n)}\mathbf{a}\}\, d\mathbf{a}\,|_{\xi=0} \tag{11}$$

for Gaussian noise backgrounds.

With continuous sampling on $(0, T)$ the a_k^* become functionals of $V(t)$, and taking the appropriate limits as $n \to \infty$ of the various quadratic forms $\Phi_k^{(n)}$, $\Psi_{kl}^{(n)}$ above[7], we obtain

$$\Psi_{kl}^{(T)} \equiv \lim_{n\to\infty} \Psi_{kl}^{(n)} = \int_{0-}^{T+} \phi_k(t)X_l(t)\, dt \tag{12a}$$

$$\Phi_k^{(T)} \equiv \lim_{n\to\infty} \Phi_k^{(n)}(\mathbf{V}) = \int_{0-}^{T+} V(t)X_k(t)\, dt, \tag{12b}$$

where X_k is the solution of

$$\int_{0-}^{T+} K_N(t, u)X_k(u)\, du = \phi_k(t), \quad (0- < t < T+). \tag{12c}$$

At this point it is convenient to choose the $\{\phi_k\}$ to obey (2), *i.e.*, to be the eigenfunctions of the noise process (5) on $(0, T)$. Then we have directly

$$X_k(t) = \phi_k(t)\lambda_k^{-1}, \quad (0 \le t \le T), \tag{13}$$

so that

$$\Psi_{kl}^{(T)} = \lambda_k^{-1}\,\delta_{kl}\,; \quad \Phi_k^{(T)} = \lambda_k^{-1}c_k(V(t)), \tag{14}$$

where c_k is given by (4). Writing $\mathbf{\Lambda}_T \equiv [\lambda_k\delta_{kl}]$, $\mathbf{c} = [c_k]$, we can express (11) now for *continuous sampling* in the case of Gaussian noise as

$$a_{kT}^*(V) = -i\frac{d}{d\xi_k} \log \int_{(\mathbf{a})} \sigma(\mathbf{a}) \cdot \exp\{\tilde{\mathbf{a}}[i\boldsymbol{\xi} + \mathbf{\Lambda}_T^{-1}\mathbf{c}] - \tfrac{1}{2}\tilde{\mathbf{a}}\mathbf{\Lambda}_T^{-1}\mathbf{a}\}\, d\mathbf{a}\,|_{\xi=0}. \tag{15}$$

Structure

Before listing results for special choices of $\sigma(\mathbf{a})$, let us remark on the structure of the optimum systems implied by (15). Letting ϕ_k be the weighting function h of a linear, relizable filter, *e.g.*,

$$h_k(T - t) \equiv \phi_k(t), \quad 0 \le t \le T, \tag{16}$$

we observe that

$$c_k(V) = \int_0^T V(t)h_k(T - t)\, dt, \tag{17}$$

so that c_k is the output of said filter at $t = T$. The operations on c_k implied by (15) can then be obtained by computation, depending on $\sigma(\mathbf{a})$, once the $\{c_k\}$ have been found. The Bayes waveform estimator with QCF here consists therefore of a parallel bank of linear, realizable filters, cf. (16), each output of which goes to a suitable computer, whose output in turn is the desired a_{kT}^*. These a_{kT}^* may then be multiplied by $\phi_k(t_\lambda)$ and added to give the Bayes estimate of the waveform at t_λ, *e.g.*,

$$S^*(t_\lambda) = \sum_{k=1}^{\infty} a_k^*\phi_k(t_\lambda).$$

By allowing t_λ to vary in $(0, T)$, the entire waveform in that interval may be reproduced (in estimate). The ϕ_k, of course, depend on the (known) spectral distribution of the background noise through (2).

[7] For a justification of the formal procedure, see Middleton, *op. cit.*, sec. (19.4-2).

Special Results

Here we shall give the Bayes estimators for some useful choices of $\sigma(\mathbf{a})$:

Case I: Gauss d.d. of $\{a_k\}$.

$$\sigma(\mathbf{a}) = K_A \exp\{-\tfrac{1}{2}(\tilde{\mathbf{a}} - \tilde{\bar{\mathbf{a}}})\mathbf{A}^{-1}(\mathbf{a} - \bar{\mathbf{a}})\}, \tag{18}$$

where $\mathbf{A} = \tilde{\mathbf{A}}$ and $\mathbf{A}$ in positive definite.

Eq. (11) yields

$$a_k^* = [(\mathbf{\Psi}^{(n)} + \mathbf{A}^{-1})^{-1}(\mathbf{\Phi}^{(n)} + \mathbf{A}^{-1}\bar{\mathbf{a}})]_k, \tag{19}$$

which for continuous sampling and the ϕ_k of (2) becomes

$$a_{kT}^* = [(\mathbf{\Lambda}_T + \mathbf{A}^{-1})^{-1}(\mathbf{\Lambda}_T \boldsymbol{c} + \mathbf{A}^{-1}\bar{\mathbf{a}})]_k. \tag{20}$$

If $\mathbf{A} = [\sigma_k^2\delta_{kl}]$, *i.e.*, the a_k are independent, (20) reduces at once to

$$a_{kT}^* = \frac{c_k + \bar{a}_k\lambda_k/\sigma_k^2}{1 + \lambda_k/\sigma_k^2}, \qquad k = 1, 2, \cdots, \tag{21}$$

which is a generalization of Glaser and Park's result, (16).[6]

Case II. Independent Rayleigh d.d. of $\{a_k\}$.

Here each a_k obeys

$$\sigma_k(a_k) = \frac{a_k \exp\{-a_k^2/2\sigma_k^2\}}{\sigma_k^2}, \qquad a_k \geq 0. \tag{22}$$

The Bayes estimators are (for continuous sampling)

$$a_{kT}^* = \frac{\pi\lambda_k}{2\sqrt{1 + \lambda_k/\sigma_k^2}} \cdot \frac{[e^{x^2}{}_1F_1(-\frac{1}{2}; 1; -x^2) + 2x/\sqrt{\pi} + (1 + 2x^2)e^{x^2}\Theta(x)]}{x\sqrt{\pi}\, e^{x^2}[1 + \Theta(x)] + 1}, \tag{23}$$

with

$$x \equiv (c_k/\sqrt{2\lambda_k})/(1 + \lambda_k/\sigma_k^2)^{\frac{1}{2}} \tag{23a}$$

where ${}_1F_1$ is the usual confluent hypergeometric function and Θ is the error integral

$$\Theta(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Case III: Independent unsymmetrical, uniform d.d. of $\{a_k\}$.

For this case

$$\sigma_k(a_k) = P_k^{-\frac{1}{2}}, \qquad 0 < a_k < P_k^{\frac{1}{2}}, \tag{24}$$

so that as $P_k \to \infty$, we have again (for continuous sampling)

$$a_{kT}^* = c_k + (2\lambda_k/\pi)^{\frac{1}{2}}e^{-x^2}[1 + \Theta(x)]^{-1}, \quad x = c_k/\sqrt{2\lambda_k}. \tag{25}$$

Note that if on the average x is large, $a_{kT}^* \simeq c_k$, which is linear in V.

Case IV: Same as III; symmetrical d.d. of $\{a_k\}$.

Now $\sigma_k(a_k)$ is modified to

$$\sigma_k(a_k) = P_k^{-\frac{1}{2}}, \qquad -P_k^{\frac{1}{2}}/2 < a_k < P_k^{\frac{1}{2}}/2, \tag{26}$$

with the result that (15) becomes simply[8]

$$a_{kT}^* = c_k. \tag{27}$$

Note that only in Cases I and IV, for independent a_k, is a_{kT}^* a linear functional of the data V.

Simple Cost Function

We consider here the "strict" version[9]

$$C(S, \gamma) = C_0\left\{A_n - \prod_{k=1}^{n} \delta(S_k - \gamma_k)\right\} \tag{28}$$

for continuous estimators. Bayes estimators here are (unconditional) maximum likelihood estimators.[1] Writing the likelihood functions

$$L_n(\mathbf{V} \mid \mathbf{a}) = \sigma(\mathbf{a})F_n(\mathbf{V} \mid \mathbf{a}); \quad l_n(\mathbf{V} \mid \mathbf{a}) = F_n(\mathbf{V} \mid \mathbf{a}) \tag{29}$$

the solutions of

$$(\text{UMLE}): \left.\frac{\partial L_n}{\partial a_k}\right|_{a_k = \hat{a}_k^*} = 0; \quad (\text{CMLE}): \left.\frac{\partial l_n}{\partial a_k}\right|_{a_k = \hat{a}_k} = 0 \tag{30}$$

yield respectively unconditional maximum likelihood estimators (UMLE) and conditional maximum likelihood (CML) estimators of the a_k. The Bayes estimator $\hat{\gamma}_\sigma^*$ of waveform S at t_λ is again given here by (7); however, the a_k^* are now replaced by $\hat{a}_k^*$. The conditional estimator $\hat{\gamma}_S$ at t_λ is likewise

$$\hat{\gamma}_S = \sum_k \hat{a}_k\phi_k(t_\lambda). \tag{30a}$$

For additive Gauss noise we readily obtain from (9)–(10b) and (30):

$$(\text{UMLE}): \left[\sigma^{-1}\frac{\partial\sigma}{\partial a_k} + \Phi_k^{(n)}(\mathbf{V}) - \sum_l^{\infty} a_l\Psi_{kl}^{(n)}\right]_{a_k = \hat{a}_k^*} = 0, \quad \text{all} \quad k, \tag{31}$$

as the condition of a UMLE of a_k. With continuous sampling and the appropriate $\{\phi_k\}$, cf. (2), (31) becomes

$$(\text{UMLE}): \left[\sigma^{-1}\frac{\partial\sigma}{\partial a_k} + (c_k - a_k)\lambda_k^{-1}\right]_{a_k = \hat{a}_{kT}^*} = 0. \tag{32}$$

The conditional MLE for (32) is at once

$$\hat{a}_{kT} = c_k(V), \tag{33}$$

e.g., just the outputs of the bank of linear filters (16), (17). For Cases I-IV considered above, we have specifically

$$\hat{a}_{kT}^*\big|_{\text{I}} = \frac{c_k + \bar{a}_k\lambda_k/\sigma_k^2}{1 + \lambda_k/\sigma_k^2} = a_{kT}^*\big|_{\text{I, }QCF} \tag{34a}$$

$$\hat{a}_{kT}^*\big|_{\text{II}} = \tfrac{1}{2}(1 + \lambda_k/\sigma_k^2)^{-1}(1 + \sqrt{c_k^2 + 4\lambda_k + 4\lambda_k^2/\sigma_k^2}) \tag{34b}$$

$$\hat{a}_{kT}^*\big|_{\text{III}} = \hat{a}_{kT}^*\big|_{\text{IV}} = c_k = \hat{a}_k\big|_{\text{III, IV}}. \tag{34c}$$

[8] Glaser and Park's result, (10).
[9] Middleton and Van Meter, *op. cit.*, cf. (4.2).

ote again that it is possible for different cost functions yield the same Bayes estimators [cf. (34a), (34c), and iddleton and Van Meter[1]]. (The associated Bayes risks, wever, are not necessarily the same.)

System Evaluation and Comparison

This is accomplished in the usual fashion by computing e (Bayes) risk for optimum systems and the average k for suboptimum ones and then comparing the results s-*à-vis* average costs, or say, input signal-to-noise ratios or the same average cost), etc. Here it is often convenient extend the measure of average risk associated with e estimate of S at t_λ to include all t_λ in $(0, T)$. We ccordingly define a time-averaged risk (TAR) by

$$R_T^* \equiv \frac{1}{T} \int_0^T R^*(t_\lambda)\, dt_\lambda \tag{35}$$

or Bayes systems), with R_T^* and $R^*(t_\lambda)$ replaced by R_T, (t_λ) for suboptimum cases. Note that since $0 \leq R^*(t_\lambda) \leq$ (t_λ), all t_λ, then $R_T^* \leq R_T$.

We illustrate these remarks with several results based n the QCF and continuous sampling. Applying (6) and), we get

$$R_T^* = \frac{C_0}{T} \int_0^T [\overline{S^2} - \overline{2S\gamma_\sigma^*} + \overline{\gamma_\sigma^{*2}}]\, dt, \tag{36a}$$

$$= \frac{C_0}{T} \sum_k [\overline{a_k^2} - \overline{2a_k a_{kT}^*} + \overline{a_{kT}^{*2}}]. \tag{36b}$$

ow

$$\sum_k \overline{a_k^2} = \int_0^T \overline{S(t)^2}\, dt = T\overline{P}_S$$

nd to determine the other averages in (36) we must pecify $\sigma(\mathbf{a})$. Here we choose Cases IV and I and find a straightforward way that

$$R_T^* \,|_{\mathrm{IV}} = \frac{C_0}{T} \sum_k \lambda_k = C_0 \psi_N, \tag{37}$$

$$R_T^* \,|_{V} = \frac{C_0}{T} \sum_k \left(\frac{\lambda_k}{1 + \lambda_k/\sigma_k^2} \right) \tag{38}$$

ince

$$\sum_k \lambda_k = \int_0^T K_N(t, t)\, dt = \psi_N T; \qquad \psi_N = \overline{N^2}, \qquad (\overline{N} = 0). \tag{38a}$$

'or Cases II and III, R_T^* is much more involved. Observe hat for white noise backgrounds, $R_T^* \,|_{\mathrm{IV}} \to \infty$, since $\lambda_k =$ $V_0/\frac{1}{2}$, all k, where W_0 = spectral intensity density of the oise: while $R_T^*|_{\mathrm{I}}$ may be finite if σ_k^2 is such that the series 38) converges.

Remarks

This approach, using a linear development of the form (1) for the signal, is a useful alternative to the direct evaluation of waveform, based on the *a priori* statistics of S, *e.g.*, $\sigma(\mathbf{S})$. The results are equivalent, because of the linear character of (1). However, it may be simpler to instrument the system when (1) is used [cf. (3)]. The same class of *a priori* knowledge as to the signal statistics is still required; here we need $\sigma(\mathbf{a})$, [cf. (8), *et seq.*]. In practice, most of the time the receivers do not know $\sigma(\mathbf{a})$ [or $\sigma(\mathbf{S})$], so that a subsidiary extremal principle must be invoked if we are to apply the decision theory approach. This is reasonable, however, and two such principles are *minimax* and *maximum average uncertainty*.[10] Both result in essentially "worst" best (*i.e.*, Bayes) systems with respect to "most unfavorable" *a priori* distributions of $\mathbf{a}$ (or $\mathbf{S}$). The uniform d.d. of Cases III and IV are minimax (with QCF), as well as yielding maximum average uncertainties with respect to maximum values.[11]

The development (1), however, does not appear to offer any advantages when S has known waveform but one or more unknown parameters, *e.g.*, $S = S(t; \boldsymbol{\theta})$ which are to be estimated. Then $a_k = a_k(\boldsymbol{\theta})$, and in general we cannot invoke independence of the a_k. Furthermore, for the simple cost functions it is not possible to obtain explicit forms of the estimators when the $\boldsymbol{\theta}$ appear transcendentally in S, *i.e.*, as a frequency, a phase, a delay, etc.

Although we have discussed interpolation (*i.e.*, filtering) only, the procedures of the above are easily extended to extrapolation (*e.g.*, prediction, etc.) if we represent the extrapolated waveform as $S(t_\lambda) = \sum_k d_k \phi_k(t_\lambda)$, t_λ outside $(0, T)$, where the d_k are random variables, now (in general) *statistically dependent* on the a_k of $S(t) = \sum_k a_k \phi_k(t)$, $(0 \leq t \leq T)$, developed on the interval $(0, T)$. The details are left to a subsequent study.

Finally, we note as an example of a case where the $\sigma(\mathbf{a})$ (or $\sigma(\mathbf{S})$) are known *a priori* that (1) may be used in communication systems where the S's are constructed at the transmitter by one or a set of random mechanisms which select the a_k, $(k = 1, \cdots m$, say), with known statistics and with given $\{\phi_k\}$. These $\{\phi_k\}$ are also known at the receiver, as are the statistics, $\sigma(\mathbf{a})$, of the a_k. The receiver then adopts an optimization principle and makes "best" estimates of the corrupted waveforms $V = S + N$ actually received, using the above information. The accuracy of such a system is being studied, as are other cost functions, *i.e.*, Bayes and Minimax risks for the maximum likelihood estimators, and the explicit comparisons of various optimum and suboptimum systems.

[10] This maximizes $-\log \sigma(\mathbf{a})$ subject to maximum value constraints on the $\mathbf{a}$, or fixed mean-square values, P_k. The former yields the independent, uniform distribution of Cases III and IV, while the latter yields the Rayleigh d. d. of Case II.

[11] Middleton, *op. cit.*, ch. 21.

Correspondence

Cumulative Distribution Functions for a Sinusoid Plus Gaussian Noise*

In certain problems which arise in applications of communication theory, it is desirable to know the probability that a variable lies within a given amplitude range. A random variable which occurs frequently in such cases is one consisting of a sinusoidal signal (with random phase) corrupted by uncorrelated additive Gaussian noise. Let $x(t)$ be such a variable:

$$x(t) = s(t) + n(t) \tag{1}$$

$$s(t) = A \sin (\omega t + \varphi) \tag{2}$$

$$n(t) = \text{Gaussian random variable} \tag{3}$$

$$E\{n(t)\} = 0 \tag{4}$$

$$E\{n^2(t)\} = \psi_0. \tag{5}$$

The phase angle, φ, is uniformly distributed between 0 and 2π radians.

In order to determine the probability that x lies between two values, a and b, it is necessary to know the cumulative probability distribution for $x(t)$ since

$$P(a \leq x \leq b) = \int_a^b p_x(x)\, dx$$
$$= P_x(b) - P_x(a) \tag{6}$$

where $p_x(x)$ is the probability density function for x and $P_x(x)$ is the cumulative probability of x

$$P_x(x) = \int_{-\infty}^x p_x(\alpha)\, d\alpha. \tag{7}$$

S. O. Rice[1] has shown that the probability density function for $x(t)$ [as defined by (1) through (5) above] is given by

$$p_x(x) = \frac{1}{\pi\sqrt{2\pi\psi_0}} \tag{8}$$
$$\cdot \int_0^\pi \exp\{-(x - A\cos\theta)^2/2\psi_0\}\, d\theta.$$

It does not appear that this result can be presented in terms of tabulated functions.

Before operating on (8) to obtain $P_x(x)$, it is convenient to transform x to a variable of unity variance

$$y = \frac{x}{\sigma_x} \tag{9}$$

$$\sigma_x^2 = E\{x^2\} = \psi_0 + \frac{A^2}{2} \tag{10}$$

so that (6) becomes

$$P(a \leq x \leq b) = P\left(\frac{a}{\sigma_x} \leq y \leq \frac{b}{\sigma_x}\right)$$
$$= P_y\left(\frac{b}{\sigma_x}\right) - P_y\left(\frac{a}{\sigma_x}\right). \tag{11}$$

It is also helpful to define a signal to noise ratio, r in the usual sense

$$r = \frac{E\{s^2\}}{E\{n^2\}} = \frac{A^2}{2\psi_0}. \tag{12}$$

* Received by the PGIT, November 24, 1958.

[1] S. O. Rice, "Mathematical Analysis of Random Noise," Bell Telephone Sys. Monograph B-1589, eq. 3.10-6, p. 105; 1945.

Cumulative Distribution Functions for the Sum of a Sinusoid and a Gaussian Variable

Values of $P_y(y) - ½$

y	$r = 0$	$r = .20$	$r = .50$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 6$	$r = 8$	$r = 10$	$r = 12$	$r = 15$	$r = 20$	$r = \infty$
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.10	0.03983	0.03958	0.03860	0.03636	0.03220	0.02936	0.02757	0.02569	0.02481	0.02431	0.02400	0.02369	0.02339	0.02253
0.20	0.07926	0.07877	0.07687	0.07255	0.06450	0.05896	0.05542	0.05164	0.04983	0.04881	0.04816	0.04754	0.04693	0.04517
0.30	0.11791	0.11721	0.11449	0.10837	0.09696	0.08901	0.08383	0.07811	0.07529	0.07369	0.07267	0.07169	0.07075	0.06804
0.40	0.15542	0.15455	0.15117	0.14363	0.12960	0.11968	0.11303	0.10538	0.10145	0.09918	0.09774	0.09635	0.09502	0.09128
0.50	0.19146	0.19046	0.18661	0.17811	0.16236	0.15104	0.14321	0.13377	0.12864	0.12560	0.12363	0.12175	0.11997	0.11503
0.60	0.22575	0.22466	0.22055	0.21159	0.19510	0.18304	0.17444	0.16352	0.15722	0.15331	0.15072	0.14822	0.14584	0.13947
0.70	0.25804	0.25692	0.25276	0.24383	0.22757	0.21552	0.20665	0.19482	0.18752	0.18275	0.17947	0.17620	0.17304	0.16482
0.80	0.28814	0.28705	0.28302	0.27459	0.25947	0.24817	0.23963	0.22767	0.21980	0.21434	0.21040	0.20630	0.20216	0.19139
0.90	0.31594	0.31490	0.31119	0.30363	0.29042	0.28054	0.27292	0.26181	0.25404	0.24833	0.24398	0.23919	0.23400	0.21958
1.00	0.34134	0.34040	0.33714	0.33074	0.32000	0.31208	0.30591	0.29664	0.28982	0.28453	0.28029	0.27533	0.26949	0.25000
1.10	0.36433	0.36352	0.36080	0.35575	0.34781	0.34219	0.33786	0.33125	0.32622	0.32214	0.31872	0.31448	0.30905	0.28367
1.20	0.38493	0.38426	0.38214	0.37851	0.37346	0.37026	0.36793	0.36448	0.36184	0.35962	0.35769	0.35517	0.35166	0.32251
1.30	0.40320	0.40268	0.40118	0.39895	0.39665	0.39576	0.39538	0.39512	0.39502	0.39495	0.39486	0.39467	0.39426	0.37120
1.40	0.41924	0.41888	0.41797	0.41703	0.41716	0.41827	0.41958	0.42207	0.42422	0.42606	0.42764	0.42966	0.43233	0.45483
1.50	0.43319	0.43298	0.43261	0.43278	0.43491	0.43757	0.44013	0.44460	0.44829	0.45139	0.45405	0.45741	0.46188	0.50000
1.60	0.44520	0.44512	0.44522	0.44630	0.44990	0.45360	0.45693	0.46245	0.46681	0.47035	0.47329	0.47689	0.48142	
1.70	0.45543	0.45547	0.45595	0.45771	0.46225	0.46648	0.47010	0.47580	0.48004	0.48331	0.48589	0.48886	0.49226	
1.80	0.46407	0.46420	0.46497	0.46719	0.47219	0.47670	0.48000	0.48520	0.48880	0.49137	0.49326	0.49527	0.49728	
1.90	0.47128	0.47149	0.47246	0.47492	0.47997	0.48402	0.48713	0.49144	0.49414	0.49591	0.49711	0.49824	0.49920	
2.00	0.47725	0.47751	0.47860	0.48113	0.48591	0.48947	0.49204	0.49532	0.49716	0.49824	0.49889	0.49943	0.49980	
2.10	0.48214	0.48242	0.48356	0.48602	0.49032	0.49328	0.49528	0.49759	0.49872	0.49931	0.49962	0.49984	0.49996	
2.20	0.48610	0.48640	0.48752	0.48981	0.49351	0.49585	0.49731	0.49883	0.49947	0.49975	0.49988	0.49996	0.49999	
2.30	0.48928	0.48958	0.49065	0.49269	0.49576	0.49753	0.49853	0.49946	0.49980	0.49992	0.49997	0.49999	0.50000	
2.40	0.49180	0.49209	0.49307	0.49485	0.49730	0.49857	0.49923	0.49977	0.49993	0.49998	0.49999	0.50000		
2.50	0.49379	0.49406	0.49493	0.49642	0.49832	0.49921	0.49962	0.49991	0.49998	0.49999	0.50000			
2.60	0.49534	0.49558	0.49634	0.49756	0.49898	0.49957	0.49982	0.49996	0.49999	0.50000				
2.70	0.49653	0.49675	0.49739	0.49837	0.49940	0.49978	0.49992	0.49999	0.50000					
2.80	0.49744	0.49763	0.49816	0.49892	0.49966	0.49989	0.49996	0.50000						
2.90	0.49813	0.49829	0.49872	0.49930	0.49981	0.49995	0.49998							
3.00	0.49865	0.49878	0.49912	0.49956	0.49990	0.49998	0.49999							
3.10	0.49903	0.49914	0.49941	0.49972	0.49994	0.49999	0.50000							
3.20	0.49931	0.49940	0.49960	0.49983	0.49997	0.50000								
3.30	0.49952	0.49958	0.49974	0.49990	0.49999									
3.40	0.49966	0.49972	0.49983	0.49994	0.49999									
3.50	0.49977	0.49981	0.49989	0.49996	0.50000									
3.60	0.49984	0.49987	0.49993	0.49998										
3.70	0.49989	0.49991	0.49996	0.49999										
3.80	0.49993	0.49994	0.49997	0.49999										
3.90	0.49995	0.49996	0.49998	0.50000										
4.00	0.49997	0.49998	0.49999											

$P_y(y) = ½ + \int_0^y p_y(y)\, dy$

$p_y(y)$ = probability density function for y

r = signal to noise ratio

$= \dfrac{\text{variance of sinusoid}}{\text{variance of Gaussian variable}}$

$E\{y^2\} = 1$

then follows by means of straightforward ansformations that

$$_{y}(y) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} [\text{erf }\{(1+r)^{1/2}y - (2r)^{1/2}\cos\theta\} - \text{erf }\{-(2r)^{1/2}\cos\theta\}]\, d\theta. \quad (13)$$

his integral has been evaluated on an 3M 704 computer using parabolic integraon with an integration interval of 2.5°. The esults are tabulated (see preceding page) $P_y(y) - 1/2$ vs y for various values of r. he accuracy of this table was checked by oubling the integration interval and reomputing the table. The results agree to least the five digits tabulated. For cometeness of the table, the cases $r = 0$ and $= \infty$ have been included although $r = 0$ elds simply the error function and $r = \infty$ orresponds to the cumulative arcsine disibution.

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n the Characteristics of Errororrecting Codes*

In his paper,[1] Sacks pointed out that in der for a code to correct e-errors, no set $2e$ or less characteristics should be nearly dependent (mod 2), and conersely. This fact and its implications may e illustrated more clearly with the aid of me basic notions of group theory and apping.

When n characteristics are written as olumn vectors of a matrix r by n, they ke the form

$$\begin{aligned} &= [a_1 a_2 a_3 \cdots a_k a_{k+1} \cdots a_n] \\ &= [a_1 a_2 a_3 \cdots a_k \mid u_r]. \end{aligned} \quad (1)$$

ecoding is accomplished by multiplying e received message I by A. Thus,

$$AI = R \quad (2)$$

here R is a column matrix of r components. rror correction is achieved by associations etween the 2^r possible versions of R and e various linear combinations of the olumn vectors of A.

As the checking bits Xr of I are deterined by satisfying the equation

$$A\begin{bmatrix} Y_k \\ X_r \end{bmatrix} \equiv 0 \qquad (\text{mod } 2) \qquad (3)$$

where Y_k represents the information bits, the code points are mapped into the null vector. Each of the 2^n possible points are mapped to some linear combination of the n-characteristics. This is, in fact, a many to one mapping, and the number of points associated to each linear combination is a multiple of the size of the code.

If a set of s characteristics are dependent on each other as

$$\sum_{i=1}^{s} a_i = 0 \qquad (\text{mod } 2), \qquad (4)$$

then we have

$$\alpha = \sum_{i=1}^{b} a_i = \sum_{j=1}^{c} a_j = \alpha'$$

$$(\text{mod } 2) \quad b + c = s. \qquad (5)$$

Thus the errors corresponding to $\alpha = \sum_{i=1}^{b} a_i$ are indistinguishable from the errors corresponding to $\alpha = \sum_{j=1}^{c} a_j$. Therefore, only one of these two sets of errors can be corrected and the other set will always slip by undetected. The alternative is to say that in this case errors can only be detected and not corrected.

Thus the problem of synthesizing a code is the same as finding a set of vectors such that this set of vectors and some of their linear combinations covers the 2^r possible nonzero vectors in a prescribed manner. A lossless code is one for which the set of vectors, and their linear combinations up to a certain number of vectors, covers the 2^r vectors in a one to one manner.

One may also visualize this situation as that of partitioning the 2^n points into 2^k geometrically similar partitions where each partition contains one code point.

The shape of the partition is determined by the n-characteristics. The lossless code is achieved when these partitions are perfect spheres. The single error correction lossless code is particularly easy to construct as one simply writes down the $2^r - 1$ nonzero vectors as the characteristics. In the general case of the Galois Fields $(p^{mr} - 1)/(p^m - 1)$ nonzero vectors are used as shown by Cocke[2].

The dependent relationships may be used to calculate the α values of Slepian directly without resulting to cosets. The method of calculation is illustrated by an example of the (8, 3) code with the characteristic matrix.

$$\begin{array}{cccccccc} a_1 & a_2 & a_3 & a_1' & a_2' & a_3' & a_4' & a_5' \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is seen that the following dependent relationships are present:

$$\begin{aligned} &a_2 + a_1' + a_2' + a_5' = 0 \\ &a_3 + a_1' + a_3' + a_5' = 0 \quad (\text{mod } 2). \; (6) \\ &a_2 + a_3 + a_2' + a_3' = 0 \end{aligned}$$

All other such relationships will involve at least five vectors. In order to increase the Slepian Q value, correcting a double error is always preferable to the correction of a triple error. Therefore, these dependent relationships can be neglected. A tabulation of double errors immediately gives the α's of the code, as shown in Table I.

TABLE I

Error Corrected	Errors Not Corrected
21′	2′5′
22′	1′5′
25′	12′
31′	3′5′
35′	1′3′
32′	2′3′
32′	23′
	33′

$\alpha_0 = 1 \quad \alpha_1 = 8 \quad \alpha_2 = 20 \quad \alpha_3 = 3$

On the other hand, there is a close relationship between the degree of dependency of the characteristics and the minimum distance of the code.

Proposition: The minimum distance of a group code is d if and only if every set of $d - 1$ vectors of the characteristic matrix is independent.

Proof: We shall prove that having a set of d linearly dependent vectors implies the existence of a pair of code points at a distance d apart and conversely.

First of all, one must show the existence of a characteristic matrix for each group code which can be shown in the following manner. Given any group code where each code is written as a column vector, it is possible to obtain a sub-matrix from this matrix that consists of k columns while the rows corresponds to information position form a unit matrix. Then the k checking position vectors are the desired characteristics. This proves the existence of a characteristic matrix for every group code.

Suppose there exists a set of d vectors of the n-characteristic matrix which is dependent. This means that given any code point if d errors corresponding to these d vectors are present, the detector will print zero. Therefore, it is clear that this message with d errors coincides with another point. So, there is at least one pair of coding points at a distance d away.

It is also clear that if the distance between some pair of code points is d, then there is at least one set of d vectors which are linearly dependent.

A lossless code which is e-error correcting is obtained if one can find a matrix where each set of $2e$ vectors is linearly independent, and if for each set of e vectors there exists at least one set of $e + 1$ vectors such that this set of $2e + 1$ vectors are dependent.

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* Received by the PGIT, April 14, 1959.

[1] G. E. Sacks, "Multiple error correcting by eans of parity checks," IRE Trans. on Informaon Theory, vol. IT-4, pp. 145–147; December, 58.

[2] J. Cocke, "Lossless symbol coding with nonprimes," IRE Trans. on Information Theory, vol. IT-5, pp. 33–34; March, 1959.

Contributors

Phillip Bello (S '52—A '55) was born in Lynn, Mass. on October 22, 1929. He received the B.S. in electrical engineering from Northeastern University, Boston, Mass., in 1953 and the S.M. degree in electrical engineering from the Massachusetts Institute of Technology, Cambridge, Mass., in 1955. His Master's thesis was concerned with characterizing the return signal in a phase comparison radar caused by a complex scintillating target. From 1955 to 1957, he was an assistant professor of Communications at Northeastern University, teaching "Transients in Linear Systems" and "Filtering and Prediction" in the graduate school. He reentered M.I.T. in September, 1958 to start the Sc.D. program in electrical engineering and is currently working on his thesis which is concerned with the application of linear transformation theory to the synthesis of linear, active and nonbilateral networks.

P. Bello

While at Northeastern University, Mr. Bello was involved in research on the IFF problem, which necessitated the use of statistical communication theory. From 1956 to 1958, he was employed by Dunn Engineering Associates, Cambridge, Mass., where he was engaged in analytical studies associated with various types of radar systems frequently involving statistical problems such as detection, and measurement of radar parameters of targets. During the summer of 1958, he was employed at the Applied Research Laboratory of Sylvania Electronic Systems, Waltham, Mass., and worked on problems in statistical radar theory.

Mr. Bello is a member of Tau Beta Pi, Sigma Xi, and Eta Kappa Nu.

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Herman Blasbalg (A '48—M '55—SM '56), was born on June 17, 1925 in Poland. He received the B.E.E. degree from the College of the City of New York, N. Y. in 1948, the M.S. degree in electrical engineering from the University of Maryland, College Park, in 1952 and the Doctor of Engineering degree from The Johns Hopkins University, Baltimore, Md., in 1956.

H. Blasbalg

From 1948 to 1951, he was employed by Melpar, Inc., Alexandria, Va. During this time he was involved in research design and development of a ppm communications system and voice channel compression as well as other applied information theory projects. From 1951 to 1956, he was employed by the Radiation Laboratory of The Johns Hopkins University where he was project supervisor of the group in charge of developing automatic airborne signal analyzer systems. Dr. Blasbalg also worked on statistical detection theory and on a mathematical theory of observation. In 1956 he became a research scientist and staff consultant. In October, 1956, he joined the staff of Electronic Communications, Inc., Baltimore, Md., and is presently involved in applied statistical decision theory, observation theory and automatic observing systems.

Dr. Blasbalg is a member of Sigma Xi and the Institute of Mathematical Statistics.

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Marvin Blum (M '56) was born on June 18, 1928 in New York, N. Y. He received the B.S. degree from Brooklyn College, N. Y., in June, 1948, and has taken graduate courses in mathematics, physics, and electrical engineering from George Washington University, American University, Maryland University, the National Bureau of Standards School, and the University of California at Los Angeles Extension.

M. Blum

He worked at the National Bureau of Standards in the Central Radio Propagation Laboratory until 1950. He then transferred to the Ordnance Division, where he conducted radar reflection studies relating to missile proximity fuzes. Since July, 1954, he has been employed at Convair, San Diego, Calif., where he is conducting theroretical investigations in smoothing and prediction filters, noise simulation, and data reduction. Presently, he is working in the newly organized Convair Astronautics Division.

Mr. Blum is a member of the Society for Applied Mathematics.

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Charles V. Freiman (M '57) was born in New York, N. Y., on June 17, 1932. He received the A.B. degree in 1954 from Columbia College, New York, N. Y., and the B.S. degree in 1955 and the M.S. degree in 1956 from the school of engineering, Columbia University, New York, N.Y.

Since 1956, he has been employed as an instructor in electrical engineering at Columbia University. He has also worked as an engineer or consultant (in logical design) for Procter and Gamble, Bell Telephone Laboratories, Electronics Research Laboratories (Columbia University), International Business Machines Corporation, and the Allard Instrument Corporation.

C. V. Freiman

At present, he is working toward the Sc.D. degree in electrical engineering at Columbia University.

Mr. Freiman is a member of Tau Beta Pi and Eta Kappa Nu.

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Janis Galejs (A '52), for a photograph and biography, please see page 131 of the September, 1958 issue of these Transactions.

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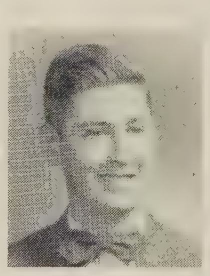
Arthur Gill was born in Haifa, Israel, on April 18, 1930. He attended the Massachusetts Institute of Technology, Cambridge, Mass., where he received the B.S. and M.S. degrees in electrical engineering in 1955 and 1956, respectively. He received the Ph.D. degree in the same field from the University of California, Berkeley, in 1959.



A. Gill

From 1954 to 1956 he served as a teaching assistant in the department of electrical engineering at M.I.T. From 1956 to 1957 he worked in the research division of the Raytheon Manufacturing Company, Waltham, Mass., where he was engaged primarily with semiconductor circuitry design. Since 1957 he has been a member of the staff of the University of California, where he has served as a teaching associate in electrical engineering and later as an assistant professor. He is also associated with the Electronic Research Laboratory of the University, where he is working on information theory problems, and with the advanced programming development group of the Bendix Computer Division of the Bendix Aviation Corporation.

Dr. Gill is a member of Eta Kappa Nu, Tau Beta Pi, and Sigma Xi.

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William A. Janos (M '59) was born on November 9, 1926, in Easton, Pa. He served in the U. S. Army from 1945–1947. In 1951

W. A. Janos

he received the B.S degree in physics from Rutgers University, New Brunswick, N. J. Since 1951 he has been with Convair and Convair-Astronautics, San Deigo, Calif., taking leaves of absence to obtain the M.A. degree in 1954 and the Ph.D egree in 1958, both in physics, from the niversity of California, Berkeley. He was ecipient of the University's appointed eaching assistantship in the physics deartment, and also the Convair Scholarship ward.

He has been engaged in applied analysis nd spectral theory related to analytical ynamics, wave propagation and diffracion, variational techniques in least-time rajectories for thrust-propelled flight, conrol system analysis and synthesis, noise heory and optimal linear estimation. He presently conducting analytical research n digital-analog communication, computaion and control systems, and initiating tudies in information theory of continuous ources.

Dr. Janos is a member of the American hysical Society.

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Julian Keilson (SM '56) was born in rooklyn, N. Y., on November 19, 1924. In 947, he received the B.S. degree in physic from Brooklyn College, N. Y., and the A.M. and Ph.D. degrees in physics from Harvard University, Cambridge, Mass., in 1948 and 1950, respectively. He was an Atomic Energy Commission predoctoral fellow from 1949 to 1950, and a research fellow in electronics at Harard University from 1950 to 1952.

J. Keilson

From 1952 to 1956, Dr. Keilson was mployed at the Massachusetts Institute f Technology's Lincoln Laboratory, Lexngton, Mass. In 1956, he joined the Applied esearch Laboratory of Sylvania Electronic ystems, Waltham, Mass., where he has orked on theoretical aspects of electronic ountermeasures, radar counter-countereasures, radar systems techniques, and most recently, the physics associated with vehicle reentry and electromagnetic propagation in the upper atmosphere. He is the author of numerous research papers on stochastic theory, diffusion in semiconductors and electromagnetic propagation.

Dr. Keilson is a member of the American Physical Society and Sigma Xi.

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Wan Hee Kim (M '56) was born in Osan, Korea, on May 24, 1926. He received the B.E. degree from the National Seoul University, Korea, in 1950. In 1953, he came to the United States after serving in the South Korean Army during the Korean War, and entered the University of Utah, Salt Lake City, under the sponsorship of Kappa Sigma Fraternity, where he received the M.S. degree in 1954 and the Ph.D. degree in electrical engineering in 1956. During 1955–1956, he was a research assistant at the University of Illinois, Urbana, where he was engaged in research in network theory.

W. H. Kim

During 1956–1957, he was employed in the I.B.M. Research Laboratory, Poughkeepsie, N. Y. Since 1957, he has been on the faculty of the department of electrical engineering, Columbia University, New York, N. Y., where he is teaching and directing research as an assistant professor in the area of network and information theory under a grant of the National Science Foundation.

Dr. Kim is a member of Tau Beta Pi and Sigma Xi.

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N. David Mermin was born in New Haven, Conn., on March 30, 1935. He received the A.B. degree from Harvard University, Cambridge, Mass., in 1956, concentrating in mathematics, and the A.M. degree in physics from Harvard in 1957. He is currently at Harvard doing research in the quantum mechanical many body problems for the Ph.D. in physics. His graduate study has been supported by National Science Foundation and General Electric predoctoral fellowships.

N. D. Mermin

During the summers of 1957 and 1958, he was employed at the Applied Research Laboratory of Sylvania Electronic Systems in Waltham, Mass.

Mr. Mermin is a member of the American Physical Society, Phi Beta Kappa, and Sigma Xi.

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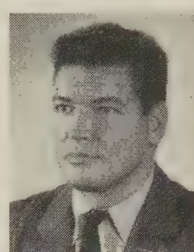
David Middleton (S '42—A '44—M '45—SM '53—F '59) was born on April 19, 1920, in New York, N. Y. He received the A. B. degree in 1942 the M. A. degree in 1945, and the Ph.D. degree in physics in 1947, all from Harvard University, Cambridge, Mass.



D. Middleton

During World War II, he was a research associate at the Harvard Radio Research Laboratory working in the field of electronic countermeasures. From 1947 to 1949, he was a research fellow in electronics at the Harvard Electronics Research Laboratory of the Division of Applied Science. In 1949, he became assistant professor of applied physics in the same division. Since 1954, he has engaged in private consulting practice with industry and the armed services. At present, his principal field of research is in statistical communication theory, including applications in electronics, electron physics, information theory, system design and evaluation, and the study of various problems in applied mathematics which are related to these fields.

Dr. Middleton is a fellow of the American Physical Society and of the American Association for the Advancement of Science, a member of the American Mathematical Society, New York Academy of Sciences, Society for Industrial and Applied Mathematics, Institute of Mathematical Statistics, Phi Beta Kappa, and Sigma Xi. He was a national Research Council predoctoral fellow in physics from 1946 to 1947, and received the National Electronics Conference Award (with W. H. Huggins), in 1956. He is the author of "Random Processes, Signals, and Noise—An Introduction to Statistical Communication Theory," to appear in McGraw-Hill's International Series in Pure and Applied Physics, December, 1959.

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The Professional Group on Information Theory is planning a special issue devoted to detection and communication by the use of matched filters. Papers are invited on the theoretical and experimental aspects of this problem and related matters. The deadline for submission of completed manuscripts is October 15, 1959. Manuscripts should be submitted to:

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